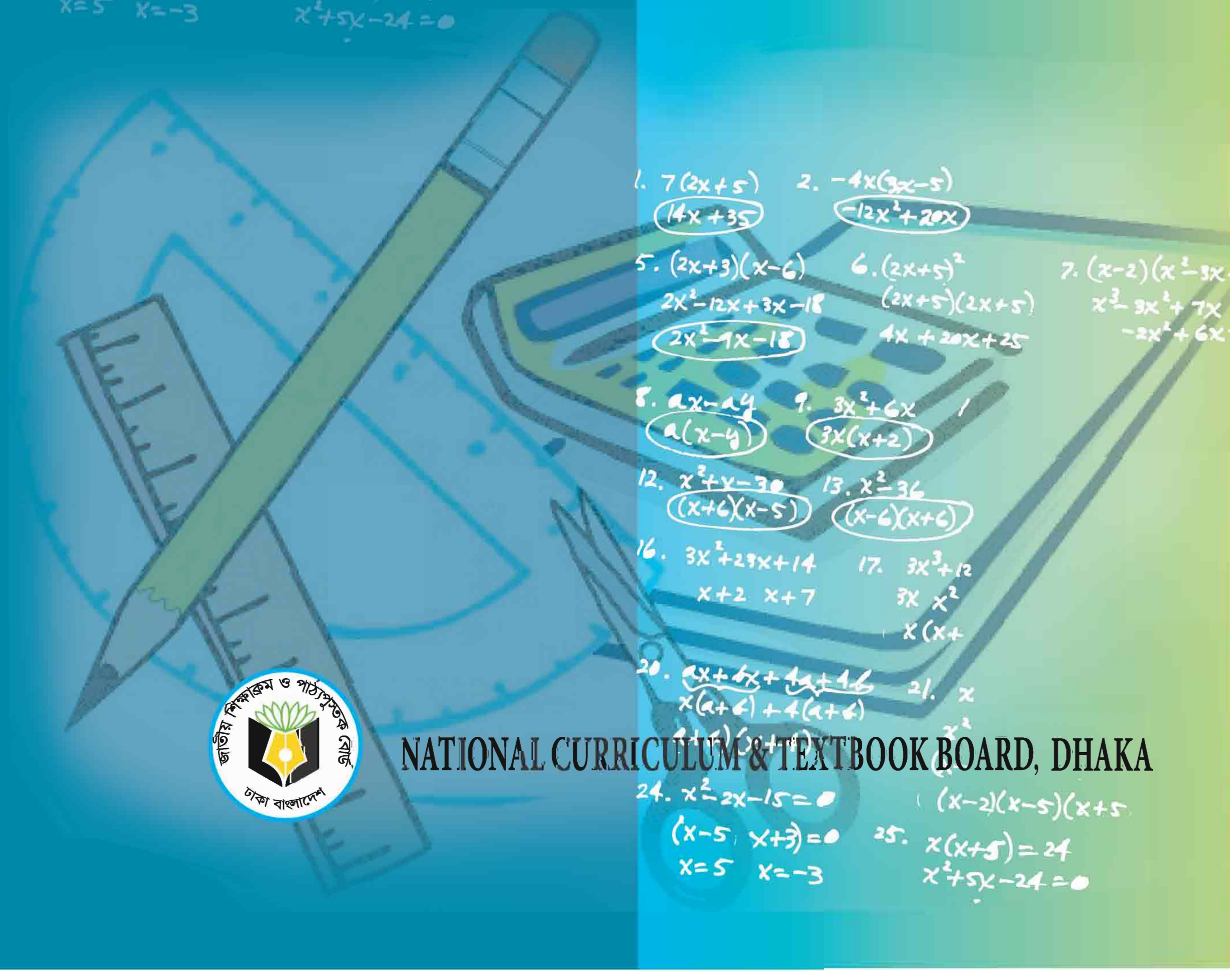
Higher Mathematics

Classes 9-10



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Higher Mathematics

Classes Nine-Ten

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Preface

Education is the pre-requisite for holistic development. In order to face the challenges of the fast changing world and to accelerate the development and prosperity of Bangladesh, there is a need for well-developed human resources. One of the most important objectives of Secondary Education is to develop students' intrinsic talents and potentials to build the country in line with the spirit of the Language Movement and the Liberation War. Besides, purpose of education at this stage is also to prepare students for higher levels of study by integrating and enhancing the basic knowledge and skills acquired at the primary level. The secondary level of education also takes into consideration the process of learning that helps students become skilled and worthy citizens in the backdrop of country's economic, social, cultural and environmental realities.

The new curriculum of secondary education has been developed keeping in mind the aims and objectives of the National Education Policy 2010. In the curriculum, national ideals, aims, objectives and demands of the time have been properly reflected. It will ensure also the learning of the students according to their age, talent and receptivity. In addition, a broad range starting from moral and human values of the students, awareness of history and culture, the Liberation War, arts-literature-heritage, nationalism, environment, religion-caste-creed and gender is given due importance. Everything is done in the curriculum to enable students to grow up a scientifically conscious nation to be able to apply science in every sphere of life and to realize the Vision of Digital Bangladesh 2021.

All textbooks are written in the light of this new curriculum. In the development of the textbooks, learners' ability, inclination aptitude and prior experience have been given due consideration. Special attention has been paid to the flourishing of creative talents of the students and for selecting and presenting the topics of the textbooks. In the beginning of every chapter, learning outcomes are added to indicate what they might learn. Various activities, creative questions and other tasks are included to make teaching-learning and assessment more creative and effective.

The subject Higher Mathematics is applied worldwide in various researches of scientific innovation. Especially, the application of Higher Mathematics in Physics, Astronomy and space research is essential. Besides, Higher Mathematics is contributing meaningfully to various researches and experiments of every day life. The learning of Higher Mathematics is very important to face the challenges of scientific world of 21st century, with all these things under consideration; the book titled higher mathematics has been introduced in the secondary level. In this case, sincere attempts have been taken to make the book easy and delightful, giving due emphasis on the understanding ability of the students.

This textbook has been written keeping in mind the promise and vision of the 21st century and in accordance with the new curriculum. So, any constructive and logical suggestions for its improvement will be paid mentionable attention. Very little time was available for writing the textbook. As a result, there could be some unintentional mistakes in it. In the next edition of the book, more care will be taken to make the book more elegant and error free.

Thanks to those who have sincerely applied their talent and labour in writing, editing, translating, drawing, setting sample questions and publishing of the book. I hope the book will ensure happy reading and expected skill acquisition of the learners.

Professor Md. Mostafa Kamaluddin
Chairman
National Curriculum and Textbook Borard, Dhaka.

CONTENTS

Chapter One
Sets and Functions

Basic concepts about sets and functions are discussed in Secondary algebra. Here we go beyond the material covered there, but also include a brief review of all pertinent concepts. In this chapter some additonal subject writters of secondary algebra have been disccussed.

After completing the chapter, the students will be able to –

- Form universal set, subset, complement of a set, power set ;
- Form union, intersection and difference of sets ;
- Prove the properties of set operations ;
- Describe equivalent sets and explain the concept of infinite set ;
- Explain the formula of determining of the union set, power sets and verify it with the help of Venn diagram and examples ;
- Solve reallife problem by using set operations and formulae ;
- Explain the concept of relations and functions by using sets ;
- Determine the domain and range of functions ;
- Explain one-one function, subjective function, injective function with examples;
- Explain inverse function.

1.1 Sets

A set is any well-defined collection of objects of the real world or of the conceptual realm. For example, the word 'mathematics, is the definite collection the letters of a, c, e, h, i, m, s, t . This is the set of the letters of the word 'mathematics' and each letter is the element of that set. We express 'set' in the capital letters of the English alphabet and by enclosing them within second braket { }, we use comma to defferentiate the elements. That is, M = {a, c, e, h, i, m, s, t}.

More examples :

(a) If $'F'$ denotes the set of the first ten non-negative integers, so :

$$F = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

(b) If D denotes the set of the days of the week, so

$D =$ {Saturday, Sunday, Monday, Tuesday, Wednesday, Thursday}
or, $D = \{x{:}x$ is the days of the week}.

Activity : Use the Roaster Method to write down
(a) the set of the months of a year ;
(b) the set of the countries of South Asia ;
(c) the set of natural numbers ;
(d) the set of Government parks in Bangladesh.

Universal Set
Consider the following sets for discussing universal set :
$P = \{x : x$ is a positive integer and $5x \leq 16\}$
$Q = \{x : x$ is a positive integer and $x^2 < 20\}$
and $R = \{x : x$ is a positive integer and $\sqrt{x} \leq 2\}$. Each element of the sets P, Q, R is required to be a positive integer.
Now consider $U = \{x : x$ is a positive integer}
So, *P*, *Q* and *R* are the subsets of U and called the universal set. A definite set is called universal set of all the sets under discussion.

Subset :
Looking at the sets $P = \{1,2,3\}, Q = \{1,2,3,4\}$ and $R = \{1,2,3,4\}$, we find that every element of P is an element of *R*; i.e., $x \in P \Rightarrow x \in R$. We say, *P* is a subset of *R* and write $P \subseteq R$. Similarly every element of the set *Q* is the element of the set *R*. i.e., $x \in Q \Rightarrow x \in R$. so *Q* is the subset of *R* aand write $Q \subseteq R$.
There is difference though the sets *P* and *Q* are the subsets of *R*. Here, mentionable that $n(P) = 3$ and $n(R) = 4$, where n(s) is the number of elements of the set *S*. So *P* is the proper subset of *R* and write $P \subset R$.
For any set A
(i) $A \subseteq A$
(ii) $\Phi \subseteq A$ (nult set Φ is the subset of any set).
If A is the subset of a finite set B. i.e., $A \subseteq B$, so $n(A) \leq n(B)$
If A is the proper subset of a finite set B. i.e., $A \subset B$, so $n(A) < n(B)$.
Note : The sign $\subseteq$ does not means a subset and $\not\subset$ does not mean proper subset.

Complement Set :
Consider $U = \{x : x$ is a positive integer} and *P* = {1, 2, 3}. If we define set $P' = \{x : 5x > 16\}$, we find its no element in set *P*. So $P' = \{4,5,6,........\}$ and it is called complement set.
Similarly, for the set *Q* = {1, 2, 3, 4}, the complement set is Q' = {5,6,7,.......}.

If U is a universal set, the complement set of P is $P' = \{x : x \notin P, x \in U\}$.

Example 1. Given that $U = \{x : x$ is an integer, $0 < x \leq 10\}$, $A = \{x : 2\times > 7\}$ and $B = \{x : 3\times < 20\}$. write down the elements of (a) set A and A' (b) set B and B' in roaster method.

Ascertain true or false of each of the statements (i) $A' \subseteq B$, (ii) $B' \subseteq A$, (iii) $A \not\subset B$

Solution : U = {1, 2, 3, 4, 5,6, 7, 8, 9, 10}

(a) $A = \{x : 2x > 7\} = \{4,5,6,7,8,9,10\}$

$A' = \{1,2,3\}$

(b) $B = \{x : 3x < 20\} = \{1,2,3,4,5,6\}$

$\therefore$ $B' = \{7,8,9,10\}$

$\therefore A' \subseteq B$ is true, $B' \not\subset A$ is false and $A \not\subset B$ is true.

Power set

Any set of all subsets of a set A, is called the power set of A and denoted by $P(A)$.

For example, if $A = \{1.2.3.\}$, so the power set of A is

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Remember : Every element of $P(A)$ is a subset of A.

Note : $B \in P(A)$ means $B \subseteq A$, If U denotes the universal set in a given context, every set appearing in that context is an element of $P(U)$.

If a set A has a finite number of elements, say n elements, its power set $P(A)$ will have 2^n elements.

Activity :

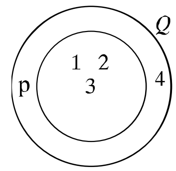
1. Given that $U = \{1,2,3,4,5,6,7,8,9,10\}$, express the following sets in roaster method :
   - (a) $A = \{x : 5x > 37\}$
   - (b) $B = \{x : x + 5 < 12\}$
   - (c) $C = \{x : 6 < 2x < 17\}$
   - (d) $D = \{x : x^2 < 37\}$
2. Given that $U = \{x : 1 \leq x \leq 20, x \in z^+\}$, express the following sets in roaster method :
   - (a) $A = \{x : x$ is a multiple of 2$\}$
   - (b) $B = \{x : x$ is a multiple of 5$\}$
   - (c) $C = \{x : x$ is a multiple of 10$\}$

   In the light of your work, ascertain the true or false of each of the statements :

   $C \subset A, B \subset A, C \subset B$
3. If, $A = \{a, b, c, d, e\}$; find $P(A)$.

Venn Diagram

The picture below illustrates the relation $P \subset Q$ which exists between the sets $P = \{1,2,3\}$ and $Q = \{1,2,3,4\}$.



A geometric picture used to illustrate the relations of more then one subsets of a set, is called a Venn Diagram. Usually a rectangle is used to represent the universal set ; circular regions inside the rectangle are used to represent subsets. Figure 1 below is a Venn diagram illustrating the universal set $U = \{x : x$ is an integer and, $0 < x \leq 10\}$, set $A = \{x : 2x > 7\}$, and $A' = \{x : 2x \leq 7\}$.

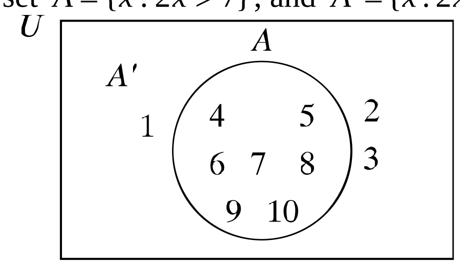


figure-1

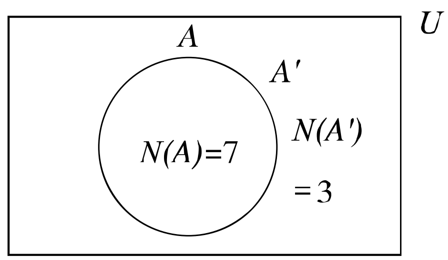


figure-2

Another Venn diagram for the same sets would be figure 2, where instead of listing the indidual elements of A or A' only the number of their elements can be written down. When we write $n(A)$, we imagine that A is a finite set.

If the universal set U is a finite set and A is any subset of U, we can write $n(A) + n(A') = n(U)$

Exampe 2. Given that $U = \{x : 2 \leq x \leq 30, x \in Z^+\}$ and $P = \{x : x$ is a factor of $30\}$.

(a) List the elements of P in roaster method.

(b) Describe the set P'

(c) Find $n(P')$.

Solution : (a) $P = \{2, 3, 5, 6, 10, 15, 30\}$

(b) $P' = \{x : x$ is not a factor of $30\}$

(c) $n(P') = n(U) - n(p)$

$= 29 - 7$

$\therefore \quad n(P') = 22$

Union of Sets

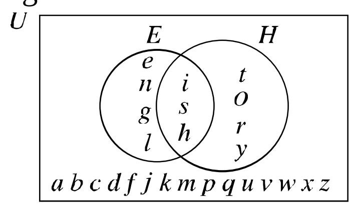
Take the set of letters of the English alphabet as the universal set U and two of its subsets respectively $E = \{e, n, g, l, i, s, h\}$ and

$H = \{h, i, s, t, o, r, y\}$

(a) Use a Venn diagram to illustrate the universal sets U, E and H.

(b) Express the elements of the set $E \cup H = \{x : x \in E$ or $x \in H\}$ in roaster method.

Solution : (a) The Venn diagram is shown below



(b) From the Venn diagram

we have $\{x : x \in E$ or, $x \in H\}$

$= \{e, n, g, l, i, s, h, t, o, r, y\}$

Note : The set $\{x : x \in E$ or $x \in H\} = \{e, n, g, l, i, s, h, t, o, r, y\}$

The set consisting of all elements of E and H is called the union of set and through the symbol $E \cup H$, it is expressed.

That is, $E \cup H = \{x : x \in E$ or $x \in H$ $\}$.

Example 3. Given the universal set $U = \{2,3,4,5,6,7,8,9\}$ and the two subsets

$A = \{x : x$ is a prime number$\}$,

$B = \{x : x$ is an odd number$\}$,

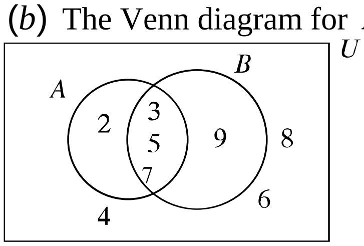
(a) List the elements of A, B and $A \cup B$.

(b) Show $A \cup B$ in a Venn diagram

(c) List the elements of set $A \cup B$ and set $A \cup B'$ in roaster method

Solution : (a) $A = \{2, 3, 5, 7\}$, $B = \{3, 5, 7, 9\}$ and $A \cup B = \{2, 3, 5, 7, 9\}$

(b) The Venn diagram for $A \cup B$ is shown below



(c) $A \cup B = \{x : x \in A$ or $x \in B\} = \{2, 3, 5, 7, 9\}$

$(A \cup B)' = \{4, 6, 8\}$

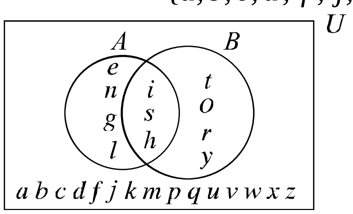
Intersection of Sets

Let us return to the subsets $A=\{e,n,g,l,i,s,h\}$ and $B=\{h,i,s,t,o,r,y\}$ of the universal set U, consisting of the letters of the English alphabet. Then $\{x : x \in A$ and $x \in B\} = \{i, s, h\}$ is the set consisting of the common elements of the sets *A* and *B*. This set is called the intersection of the sets *A* and *B* and denoted by $A \cap B$. That is,

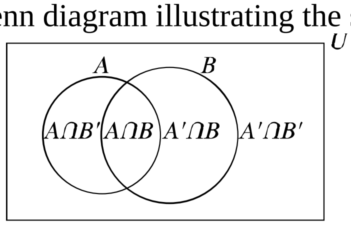
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Similarly we get,

$$A \cap B' = \{x : x \in A \text{ and } x \in B'\} = \{e, n, g, l\};$$
$$A' \cap B = \{x : x \in A' \text{ and } x \in B\} = \{t, o, r, y\};$$
$$A' \cap B' = \{x : x \in A' \text{ and } x \in B'\}$$
$$= \{a, b, c, d, f, j, k, m, p, q, u, v, w, x, z\}.$$



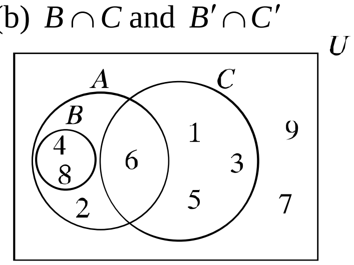
A Venn diagram illustrating the sets above is shown below :



Example 4. Given that $U = \{1,2,3,4,5,6,7,8,9\}$, $A = \{2, 4, 6, 8\}$, $B = \{4, 8\}$ and $C = \{1, 3, 5, 6\}$, draw the Venn diagram ;

(a) $A \cap B$ and $A \cap B'$

(b) $B \cap C$ and $B' \cap C'$



Solution : (*a*) When $B \subseteq A$

$$A \cap B = B = \{4,8\}$$
$$A \cap B' = A = \{2,6\}$$

(b) $B \cap C = \{\ \}$

$$B' \cap C' = B = \{2,7,9\}$$

From those examples, we get, $B \cap C = \{\ \}$, therefore, sets B and C are called disjoint sets. So A and B are disjoint sets $\Leftrightarrow A \cap B = \phi$.

Example 5. Given that $U = \{p, q, r, s, t, u, v, w\}$, $A = \{p, q, r, s\}$, $B = \{r, s, t\}$ and $C = \{s, t, u, v, w\}$.

(a) Write down the elements of $A \cap B, B \cap C$ and $C \cap A$ in roaster method and show in a Venn diagram.

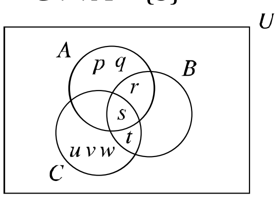
(b) Express the elements of $A \cap B \cap C$ in roaster method.

Solution :

(a) $A \cap B = \{r, s\}$

$B \cap C = \{r, t\}$

$C \cap A = \{s\}$



(b) $A \cap B \cap C = \{r, s\} \cap C = \{r.s\} \cap \{s.t.u.v, w\}$

$= \{s\}$

Example 6. Given a universal set U and $A \cap B = \phi$, shade the following sets in separate Venn diagram :

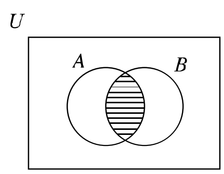
(a) $A \cap B$ (b) $A' \cap B$

(c) $A \cap B'$ (d) $A' \cap B'$

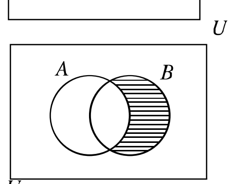
Show that, $n(A \cap B + n(A' \cap B) + n(A \cap B') + n(A' \cap B' = n(U)$

Solution :

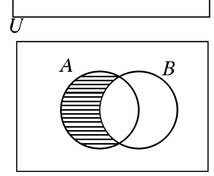
(a) $A \cap B$



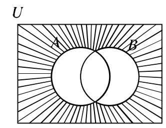
(b) $A' \cap B$



(c) $A \cap B'$



(d) $A' \cap B'$

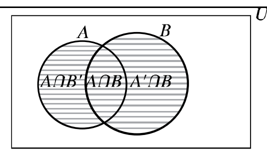


In the Venn diagram, the number of elements of each subset of the universal set U has been shown and from these we get,

$n(A \cap B + n(A' \cap B) + n(A \cap B') + n(A' \cap B' = n(U)$

In the case of any two subsets of a universal set U, it is written down as

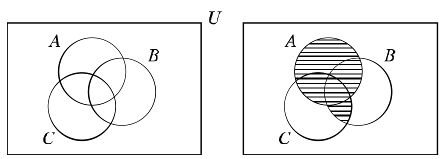
$$n(A \cap B + n(A' \cap B) + n(A \cap B') + n(A' \cap B' = n(U)$$



Example 7. Show in Venn diagram by shading

(a) $A \cap (B \cup C)$ (b) $A \cup (B \cap C)$

Solution :



Example 8. Given that $U = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{x : x$ is an even number$\}$, and $B = \{x : 7 < 3x < 25\}$

(a) Write down the elements of A, B, $A \cap B$, $A \cup B$ and $A \cap B'$ in roaster method.

(b) Determaine the elements x such that $x \in A$ and $x \notin B$

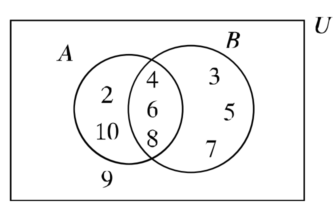
(c) Determine the elements x such that $x \notin A$ and $x \notin B$.

Solution : (a) $A = \{2,4,6,8,10\}$ $A \cap B = \{4,6,8\}$

$B = \{3,4,5,6,7,8)$

$A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10\}$

$A \cap B' = \{2,10\}$



(b) $x \in A$ and $x \notin B$

$\Leftrightarrow x \in A$ and $x \in B'$

$\Leftrightarrow x \in A \cap B'$

$\therefore x = 2, 10$

(c) $x \notin A$ and $x \notin B$

$\Leftrightarrow$ $x \in A'$ and $x \in B'$

$\Leftrightarrow$ $x \in A' \cap B' = \{9\}$

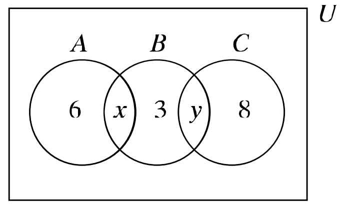
$\therefore$ $x = 9$

Example 9. In the Venn diagram, the number of elements of each subset of the universal set U has been shown. Here it is noted that $U = A \cup B \cup C$.

(a) Given that $n(B) = n(C)$; find the value of x.

(b) Given that $n(B \cap C) = n(A \cup B')$; find the value of y.

(c) What is $n(U)$?



Solution : $(a) \quad n(B) = n(C)$

$$x + 3 + y = y + 8$$

$$x = 5$$

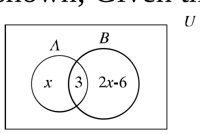
(b) $\quad n(B \cap C) = n(A \cup B')$

$$y = 6$$

(c) $n(U) = 6 + x + 3 + y + 8$

Activity :

1. Given that $U = \{1, 2, 3, 4, 5, 6, 7, 8,9\}$ and $A = \{x : x$ is multiple of 3},
   show that
   $(a) \quad A \cup A' = U \qquad (b) \quad A \cap A' = \phi$
2. Given that $U = \{3, 4, 5, 6, 7, 8, 9\}$, $A = \{x : x$ is a prime number} and $B = \{ x : x$ is an even number}. List the elements of the sets A, and $A \cap B$ in roaster method and show in a Venn diagram. Show that,
   $(a) \; A' \cap B' = \{9\} \qquad (b) \quad A \subseteq B'$ and $A \subseteq A'$.
3. Refer to the Venn diagram the elements of A the sets A and B have been shown; Given that, $n(A) = n(A' \cap B)$, find

   (a) The value of x .
   (b) the values of $n(A)$ and $n(B)$
4. Given that $U = \{p, q, r, s, t, u, y, w\}$, $A = \{p, q, r, s\}$
   $B = \{r, s, t\}$ and $C = \{s, t, u, v, w\}$
   $(a) \; n(A \cup B)$ = What ?
   (b) List the elements of $(A \cup B)'$ and $A \cup B \cup C$.
5. Show by shading in a Venn diagram : (a) $(P \cap Q) \cap R'$ (b) $(A \cap B') \cup C$

Properties of Set Operations

So far we have discussed union, intersection and disjoint of sets. Now we shall discuss the properties of these operations.

Properties of Union and Intersection

Proposition 1. Commutative law

Suppose, A = {1, 2, 4} and B = {2, 3, 5} are two set. so,

$A \cup B$ = {1, 2, 4} $\cup$ {2, 3, 5} As $A \cup B = \{x : x \in A \text{ or } x \in B\}$

= {1, 2, 4, 3, 5}

$B \cup A = \{2,3,5\} \cup \{1,2,4\}$

= {2, 3, 5, 1, 4}

As $A \cup B$ and $B \cup A$ have the same element,

Therefore, $A \cup B = B \cup A$

Similarly, taking $A = \{a, b, c\}$ and $B = \{b, c, a\}$, it can be shown that $A \cup B = B \cup A$

Generaly for any two sets A and B, it is shown

$$\boxed{A \cup B = B \cup A}$$

This is the commutative law of union of sets.

Therefore, union of sets conveys the commutative law.

Note : Similarly of the commutative law.

$$\boxed{A \cap B = B \cap A}$$

Proposition 2. Associative law

Use Venn Diagram to make the law clear. Let, A, B and C are three sets.

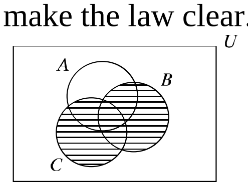


figure a (i)

Shaded Area of $B \cup C$

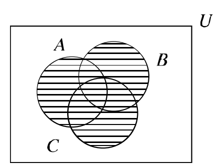


figure a (ii)

Shaded Area of $A \cup (B \cup C)$

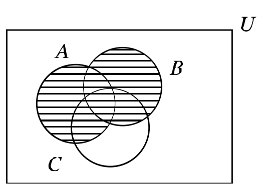


figure b (i)

Shaded Area of $A \cup B$

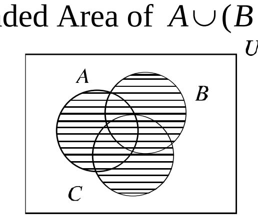


figure b (ii)

Shaded Area of $(A \cup B) \cup C$

It is clear from Venn Diagram a (ii) and b (ii) that, $(A \cup (B) \cup C$ = $(A \cup B) \cup C$

Try to understand this rule by taking three sets $A = \{a, b, c, d\}$, $B = \{b, c, f\}$ and $C = \{c, d, g\}$.

Here, $B \cup C = \{b, c, f\} \cup \{c, d, g\}$

$= \{b, c, f, d, g\}$.

And $A \cup (B \cup C) = \{a, b, c, d\} \cup \{b, c, f, d, g\}$

$= \{a, b, c, d, f, g\}$.....................(i)

Now, $A \cup B = \{a, b, c, d\} \cup \{b, c, f\}$

$= \{a, b, c, d, f\}$

And $(A \cup B) \cup C = \{a, b, c, d, f\} \cup \{c, d, g\}$

$= \{a, b, c, d, f, g\}$.....................(ii)

From (i) and (ii), we get, $A \cup (B \cup C) = (A \cup B) \cup C$

Generally for any three sets *A*, *B* and *C*.

$$\boxed{A \cup (B \cup C) = (A \cup B) \cup C}$$

$\therefore$ Union of sets conveys the associative law.

Similarly, intersection of sets also conveys the associative law.

i.e., $A \cap (B \cap C) = (A \cap B) \cap C$

Proposition 3. For $A \cup A$: Let $A = \{2, 3, 5\}$

$A \cup A = \{2, 3, 5\} \cup \{2, 3, 5\}$

$= \{2, 3, 5\}$

$= A$

In the same way, it can be shown $A \cup A = A$ by taking $A = \{x, y, z\}$,

Decision : For any set A

$$\boxed{A \cup \mathrm{A} = \mathrm{A}}$$

Do yourself in the same way : $A \cap A = A$

Proposition 4. If $A \subset B$, so $A \cup B = B$.

Let, A = {1, 2, 3} and $B = \{x : x \in N, 1 \leq x \leq 5\}$ are two sets

$\therefore$ $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$

$\therefore$ $A \subset B$.

Now, $A \cup B = \{1, 2, 3\} \cup \{1, 2, 3, 4, 5\}$

$= \{1, 2, 3, 4, 5\}$

$= B$.

Thus, if $A \subset B$, so $A \cup B = B$ and if $B \subset A$, so $A \cup B = A$.

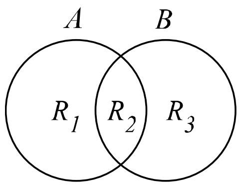
Do yourself in the same way : If $A \subset B$, so $A \cap B = A$ and if $B \subset A$, os $A \cap B = B$

Proposition 5. $A \subset (A \cup B)$. Let, *A* and *B* are two sets.

Observe the adjoining figure. The regions R_1 and R_2 are in set *A*.

Again, the regions R_2 and R_3 are in set *B*. So, the regions R_1 and R_2 are in the regions R_1, R_2 and R_3

i.e., $A \subset (A \cup B)$.



Proof : For any set *A* and *B*:

$\boxed{A \subset (A \cup B)}$ and $\boxed{B \subset (A \cup B)}$

Note : Do yourself in the same way : For any set *A* and *B*, $(A \cap B) \subset A$ and $(A \cap B) \subset B$

Proposition 6 : For $A \cup U$ and $A \cup \phi$ = *A*. We know that, $A \subset U$ and $\phi \subset A$, by (4) $A \cup U = U$ and $A \cup \phi = A$.

Activity :

1. Find $A \cup B$ where $A = \{x | x$ is an integer, $-2 \le x < 1\}$ and $B = \{x | x$ is a prime number, $24 \le x \le 28\}$
2. Find $A \cup U$ where $U = \{x | x$ is an integer, $-2 < x < 3\}$ and $A = \{x | x \in z, -1 < x \le 1\}$
3. If $A = \{2,3,5\}, B = \{a,b,c\}, C = \{2,3,5,7\}$ and $D = \{a,b,c,d\}$, prove that $(A \cup B) \subset (C \cup D)$
4. If $A = \{a, b, c\}$ and $B = \{b, c, d\}$, verify $A \cap B = B \cap A$.
5. If $A = \{1,3,5,7\}$, $B = \{3,7,8\}$ and $C = \{7,8,9\}$, show that $(A \cap B) \cap C = (B \cap C) \cap A$.

Proposition 7. Distributive Law : For any set A, B, C, show that

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof : (a) Suppose, $x \in A \cup (B \cap C)$

Then $x \in A$ or, $x \in B \cap C$

$\Rightarrow x \in A$ or, ($x \in B$ and $x \in C$)

$\Rightarrow$ ($x \in A$ or, $x \in B$) and ($x \in A$ or, $x \in C$)

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore A \cup (A \cap B) \subset (A \cup B) \cap (A \cup C)$ .... (i)

Again suppose, $x \in (A \cup B) \cap (A \cup C)$

Then, $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow (x \in A$ or, $x \in B)$ and $(x \in A$ or, $x \in C)$

$\Rightarrow x \in A$ or, $(x \in B$ and $x \in C)$

$\Rightarrow x \in A$ or, $x \in B \cap C$

$\Rightarrow x \in A \cup (B \cap C)$

$\therefore (A \cup B) \cap (A \cup C) \subset (A \cup (B \cap C)$ .... (ii)

Therefore, from (i) and (ii) we get, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) Do yourself in the similar way.

Activity :

(i) Verify the distributive laws for the sets

A = {1, 2, 3, 6}, *B* = {2, 3, 4, 5} and C = {3, 5, 6, 7} (ii) show the proof in a each Venn diagram.

Decision : each of the operations of union and intersection of sets is distributive with respect to the other operation.

Proposition 8. DeMorgans Law

For any subsets *A* and *B* of an universal set U, we have

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

Proof (a) : Suppose, $x \in (A \cup B)'$

Then, $x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

$\therefore \ (A \cup B)' \subset A' \cap B'$

Again, suppose, $x \in A' \cup B'$

Then, $x \in A'$ or, $x \in B'$

$\Rightarrow x \notin A$ or, $x \notin B \Rightarrow x \notin A \cup B$

$\Rightarrow x \in (A \cup B)'$

$\therefore A' \cap B' \subset (A \cup B)'$

Therefore, $(A \cup B)' = A' \cap B'$ (proved).

(b) Do yourself in similar way.

Proposition 9. For any subset *A* and *B* of a universal set U we have $A \setminus B = A \cap B'$

Proof : Suppose $x \in A \setminus B$

Then, $x \in A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \in B'$

$\therefore\ x \in A \cap B'$

$\therefore A \setminus B \subset A \cap B'$

Again suppose $x \in A \cup B'$

$\Rightarrow x \in A$ and $x \in B'$

$\Rightarrow x \in A$ and $x \notin B$

$\therefore\ x \in A \setminus B$

$\therefore\ A \cap B' \subset A \setminus B$

Therefore, $A \setminus B = A \cap B'$

Proposition 10. For any set A, B, C we have,

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof : (a) By definition

$A \times (B \cap C)$

$= \{(x, y) : x \in A, y \in B \cap C\}$

$= \{(x, y) : x \in A, y \in B$ and $y \in C\}$

$= \{(x, y) : (x, y) \in A \times B$ and $(x, y) \in A \times C\} = \{(x, y) : (x, y) \in (A \times B) \cap (A \times C)$

$A \times (B \cap C) \subset (A \times B) \cap (A \times C)$

Again, $(A \times B) \cap (A \times C)$

$= \{(x, y) : (x, y) \in A \times B$ and $(x, y) \in A \times C\}$

$= \{(x, y) : x \in A, y \in B$ and $x \in A, y \in C\}$

$= \{(x, y) : x \in A, y \in B \cap C\}$

$= \{(x, y) : (x, y) \in A \times B \cap C\}$

$\therefore\ (A \times B) \cap (A \times C) \subset A \times (B \cap C)$

i.e., $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(b) Do yourself in similar way.

11. Som more propositions of the method of sets

(a) $A \subset A$ holds for every set *A*.

(b) The empty set ϕ is a subset of any set *A*.

(c) *A* = *B* holds if and only if $A \subset B$ and $B \subset A$.

(d) If $A \subset \phi$, so $A = \phi$.

(e) If $A \subset B$ and $B \subset C$, so $A \subset C$

(f) For any set *A* and *B*, $A \cap B \subset A$ and $A \cap B \subset B$ hold.

(g) For any set *A* and *B*, $A \subset A \cup B$ and $B \subset A \cup B$ hold.

Proof : (b) : Let $\phi \notin A$, so by definition there is such *x* that $x \in \phi$. But $\phi \notin A$ as there is not at all any element in empty set.

$\therefore$ $\phi \notin A$ is not true

$\therefore$ $\phi \in A$

(d) Given that $A \subset \phi$, again we know $\phi \subset A$, so $A = \phi$ [from propositon c]

(g) Then $A \cup B$ consist of the common elements and all elements of *A*. So, as per the definition of a subset of $A \subset A \cup B$. By the same logic $B \subset A \cup B$.

Note : Do yourself the proposition (*a*) , (*c*), (*e*) and (*f*).

Activity : Where necessary all sets are to be considered as subsets of a universal set U .

1. Show that :
   $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$
2. Show that $A \subset B$ holds if and only if any one of the following condition holds :
   (a) $A \cap B = A$
   (b) $A \cup B = B$
   (c) $B' \subset A$
   (d) $A \cap B' = \phi$
   (e) $B \cup A' = U$
3. Show that,
   (a) $A \setminus B \subset A \cup B$
   (b) $A' \setminus B' = B \setminus A$
   (c) $A \setminus B \subset A$
   (d) $A \subset B$, then $A \cup (B \setminus A) = B$.
   (e) $A \cap B = \phi$, then $A \subset B'$ and $A \cap B' = A$ and $A \cup B' = B'$
4. Show that,
   (a) $(A \cap B)' = A' \cup B'$
   (b) $(A \cup B \cup C)' = A' \cap B' \cap C'$
   (c) $(A \cap B \cap C)' = A' \cup B' \cup C'$

Equivalent and Infinite Sets

One One Correspondence

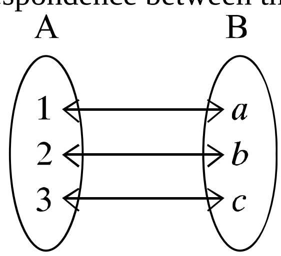
Let $A = \{a, b, c\}$ be the set of three persons and $B = \{30, 40, 50\}$ be the set of their ages. Moreover, suppose, the age of *a* is 30, the age of *b* is 40 and the age *c* is 50. So, it is said tha there is an one-one correspondence between the sets *A* and *B*.

Definition : If the matching of an element of *B* with every element of *A* and an element of *A* with every element of *B* are established, such matching is called an one one correspondence.

We write $A \leftrightarrow B$ to indicate that there exists an one-one correspondence between the sets *A* and *B* and if any element *x* of the set *A* corresponds to the element *y* of the set *B*, we write $x \leftrightarrow y$.

Equivelent sets

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ be two sets. By establishing an one one correspondence between the sets *A* and *B*, we show it in the figure below :

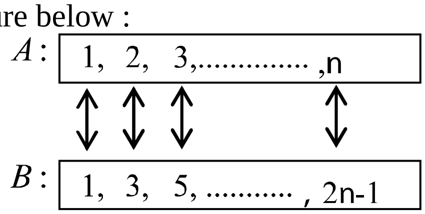


Definition : Two sets *A* and *B* are said to be equivalent if one can extablish an one-one correspondence $A \leftrightarrow B$ between the sets *A* and *B*.

We write $A \sim B$ to indicate that, A and B are equivalent sets. If $A \sim B$ is symbol, any one of them is called equivalent to the other.

Example 10. Shw that $A = \{1, 2, 3, ... n\}$ and $B = \{1, 3, 5, ...2n-1\}$ are equivalent sets, where n is any natural number.

Solution : We exhibit an one-one correspondence between the sets *A* and *B* in the picture below :



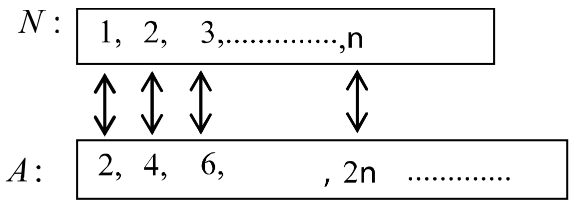
So the sets *A* and *B* are equivalent.

Remark : The one-one correspondence indicated above is described

by $A \leftrightarrow B :$ $k \leftrightarrow 2k-1,\ k \in A$

Example 11. Show that N, the set of natural numbers and $A = \{2, 4, 6, \ldots\ n \ldots\}$, the set of even natural numbers, are equivalent.

Solution : Here, $N = \{1, 2, 3, \ldots\ n \ldots\}$. We exhibit an one-one correspondence between the sets *N* and *A* in the picture below :

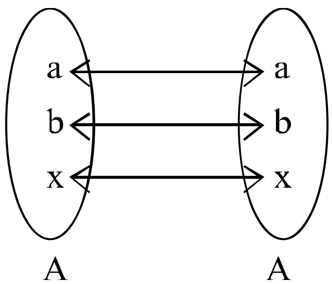


Therefore, *N* and *A* are equivalent sets.

Remark : The one-one correspondence indicated above, is described by

$N \leftrightarrow A : n \leftrightarrow 2n,\ n \in N$.

Note : Empty set ϕ is supposed to be equivalent to itself. i.e., $\phi \sim \phi$.



Proposition 1. Every set *A* is equivalent to itself.

Proof : If $A \sim \phi$, it is taken $A \sim A$.

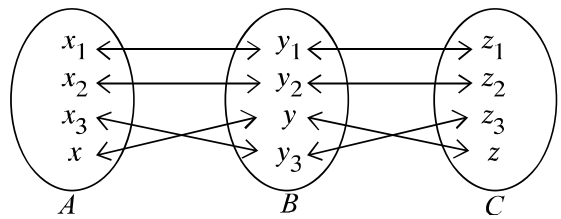
Suppose $A \neq \phi$

If every elements of *A* make a one-one correspondence by itselt, these established a one-one corrspondence i.e., $A \leftrightarrow A : x \leftrightarrow x, x \in A$.

Therefore, $A \sim A$

Proposition 2 : If *A*, *B* are equivalent sets and *B*, *C* are equivalent sets, then the sets *A*, *C* are equivalent.

Proof : Since $A \sim B$, we can associate every element *x* of *A* with a unique element *y* of *B*. Again, since $B \sim C$, we can associate with that element y of *B* a unique element z of *C*. So we can associate the element *x* of *A* with the unique elements z of *C*. This association of the elements of *A* with the elements of *C* is an one-one correspondence. i.e., $A \sim c$



Finite and Infinite sets

Counting the elements of the set $A = \{15, 16, 17, 18, 19, 20, 21, 22\}$, we find that A has 8 elements. This counting is completed by establishing an one-one correspondence of the set A with the set $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$:

$$\begin{array}{ll} A = & \{15, \ 16, \ 17, \ 18, \ 19, \ 20, \ 21, \ 22\} \\ & \ \ \updownarrow \ \ \ \ \updownarrow \ \ \ \ \updownarrow \ \ \ \ \updownarrow \ \ \ \ \updownarrow \ \ \ \ \updownarrow \ \ \ \ \updownarrow \ \ \ \ \updownarrow \\ \text{B} = & \{ 1, \ \ 2, \ \ 3, \ \ 4, \ \ 5, \ \ 6, \ \ 7, \ \ 8\} \end{array}$$

A set whose elements can be fixed by such counting, is called a finite set. an empty set is also treated as a finite set.

Definition : (a) The empty set ϕ is treated as a finite set, the number of its elements being 0.

(b) If any set A and $J_m = \{1, 2, 3, ..., m\}$ are equivalent where $m \in N$, so A is a finite set and number of elements of A is m.

(c) If A is a finite set, the number of element of A is denoted by $n(A)$.

(d) If any set A is not finite, it is called infinite set.

Note 1 : Each of $J_1 = \{1\}, J_2 = \{1, 2\}, J_3 = \{1, 2, 3\}$ etc. is the finite subset of N and $n(J_1) = 1, \ n(J_2) = 2, \ n(J_3) = 3$ etc.

Really $J_m \sim J_m$ (see proposition 1 of this chapter) and $n(J_m) = m$.

Note 2. Only the number of elments of a finite set is to be fixed. So notation $n(A)$ implies that the set A is finite.

Note 3. If A and B are equivalent sets and if one of these sets is finite, the other set is finite and $n(A) = n(B)$ holds.

Proposition 3. If A is a finite set and B is a proper subset of A, so B too is a finite set and $n(B) < n(A)$ holds.

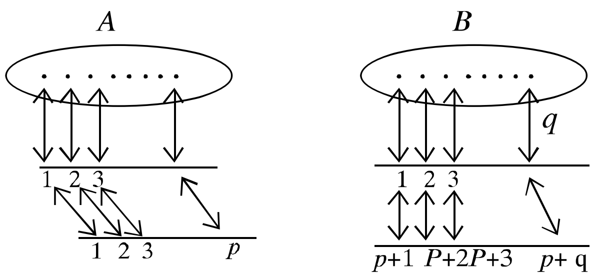
Proposition 4. The set A is infinited if and only if there exists A and a proper subset equivalent to A.

Note 5 : N is an infinite set (see example 11).

Number of Element of Finite Sets

Elements of finite set A is denoted by $n(A)$ and explain how to determine $n(A)$.

Suppose, $n(A) = P > 0, n(B) = q > 0,$ where $A \cap B = \phi$



From the above mentioned one-one correspondence, we see , $A \cup B \sim J_{p+q}$

i.e., $n(A \cup B) = p + q = n(A) + n(B)$.

Proposition 1. If A and B are disjoint set, so $n(A \cup B) = n(A) + n(B)$
expanding the proposition, we can say, $n(A \cup B \cup C) = n(A) + n(B) + n(C)$
$n(A \cup B \cup C \cup D) = n(A) + n(B) + n(C) + n(D)$ etc.
Where, the sets A, B, C, D are disjoint to each other.

Proposition 2. For any finite set A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Proof : Here, $A \setminus B, A \cap B$ and $B \setminus A$ are disjoint sets to each either see (Venn diagram) and

$A = (A \setminus B) \cup (A \cap B)$

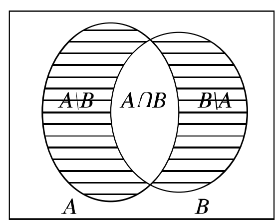
$B = (B \setminus A) \cup (A \cap B)$

$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$

$\therefore n(A) = n(A \setminus B) + n(A \cap B) .....(i)$

$n(B) = n(B \setminus A) + n(A \cap B) ....(ii)$

$n(A \cup B) = n(A \setminus B) + n(A \cap B) + n(B \setminus A) ......(iii)$



Therefore, from (i) we get, $n(A \setminus B) = n(A) - n(A \cap B)$
From (ii) we get, $n(B \setminus A) = n(B) - n(A \cap B)$
Now, $n(A \setminus B)$ and $n(B \setminus A)$ putting in (iii) we get,

$$n(A \cup B) = n(A) - n(A \cap B) + n(B) - n(A \cap B) + n(A \cap B)$$

$$\therefore \; n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Activity :

1. In each of the following cases, describe all possible one-one correspondence between the sets A an B :
   (a) $A = \{a, b\}$ $\quad B = \{1, 2\}$.
   (b) $A = \{a, b, c\}$ $\quad B = \{a, b, c\}$
2. For each one-one coresspndence described in the above question, describe the set $F = \{(x, y) : x \in A, y \in B\}$ and $x \leftrightarrow y$ in the roaster method.

3.	Let $A=\{a,b,c,d\}$ and B = {1, 2, 3,4}. Describe a subset F of $A \times B$ such that associating the second component of each ordered pair in that subset with its first component, yields an one-one correspondence of A and B in which $a \leftrightarrow 3$.
4.	Show that the set, $A=\{1,2,3,........n\}$ and $B=\{1,2,2^2,..........2^{m+1}\}$ are equivalent.
5.	Show that the set, $S=\{3^n : n=0 \text{ or } n \in N\}$ is equivalent to N.
6.	Describe a proper subset of the above set S which is equivalent to S.
7.	Show that the set $A=\{1, 3, 5, 7,.....\}$ of all odd natural numbers is an infinite set.

Power Set

In secondary algebra, the power set has been discussed elaborately. Only the examples of the power set have been placed here.

Example 12. If A = {1, 2, 3} and B = (2, 3, 4}, show that
$P(A) \cap P(B) = P(A \cap B)$.

Solution : Here, A = {1, 2, 3} and B = (2, 3, 4}

Therefore, $P(A)=\{\phi,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},(1,2,3\}\}$

And $P(B)=\{\phi,\{2\},\{3\},\{4\},\{2,3\},\{2,4\},\{3,4\},(2,3,4\}\}$

$\therefore$ $P(A) \cap P(B)=\{\phi,\{2\},\{3\},\{2, 3\}\}$

Now, $A \cap B=\{1,2,3\} \cap \{2, 3,4\}$

$= \{2, 3\}$

$\therefore$ $P(A \cap B)=\{\phi,\{2\},\{3\},\{2, 3\}\}$

Therefore, $P(A) \cap P(B) = P(A \cap B)$.

Example 13. If $A=\{a,b\}$ and $B=\{b,c\}$, show that $P(A) \cup P(B) \subset P(A \cap B)$

Solution : Here, $P(A)=\{\phi,\{a\},\{b\},\{a, b\}\}$

$P(B)=\{\phi,\{b\},\{c\},\{b, c\}\}$

$\therefore$ $P(A) \cup P(B)=\{\phi,\{a\},\{b\},\{c\},\{a,b\},\{b, c\}\}$

Again, $A \cup B=\{a,b, c\}$

$\therefore$ $P(A \cup B)=\{\phi,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,b,c\}\}$

Therefore, $P(A) \cup P(B) \subset P(A \cap B)$.

Activity :

1. If A = {1, 2, 3}, B = {1, 2}, C = {2, 3} and D = {1, 3}, show that
   $P(A)=(A,B,C,D,\{1\},\{2\},\phi\}$.
2. If A = {1, 2} and B = {2, 5}, show that
   $P(A)=\{(A,B,C,D,\{1\},\{2\},\{3),\phi\}$.

$(i)\ P(A) \cap P(B) = P(A \cap B)$
$(ii)\ P(A) \cup P(B) \neq P(A \cup B)$.

Use of Sets in Real Life Problems :

Venn diagram is used in solving real life problems. Here it is noted that the elements of each set which will be written down in Venn diagram, is shown through some examples.

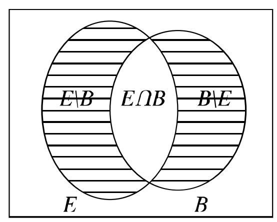
Example 14. Out of 50 persons, 35 can speak English, 25 can speak english and Bangla ; every one can speak at least one of these two languages. How many persons can speak Bangla, how many parsons can speak only Bangla ?

Solution : Let S be the set of the 50 persons, E be the set of those persons among them who can speak English, B be the set of persons who can speak Bangla.

We are given that $n(S) = 50$, $n(E) = 35$, $n(E \cap B) = 25$ and $S = E \cup B$

Let, $n(B) = x$

Then from, $n(S) = n(E \cup B) = n(E) + n(B) - n(E \cap B)$, we get, then as per questions $50 = 35 + x - 25$



or, $x = 50 - 35 + 25 = 40$

or, $n(B) = 40$

$\therefore$ 40 persons can speak Bangla.

Now, the set of those who can speak only Bangla is $(B \setminus E)$.

Let, $n(B \setminus E) = y$. Since the sets $E \cap B$ and $B \setminus E$ are disjoint, and $B = (E \cap B \cup (B \setminus E)$ [see Venn diagram]

Therefore, $n(B) = n(E \cap B) + n(B \setminus E)$

$\therefore$ 40 = 25 + y

or, $y = 40 - 25 = 15$

i.e., $n(B \setminus E) = 15$

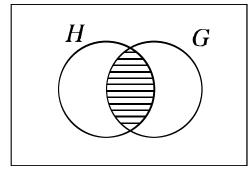
$\therefore$ 15 persons can speak only Bangla.

Thus, 40 persons can speak Bangla and 15 persons can speak only Bangla.

Example 15. If the sets are G and H respectively of the students who study Geography and History, answer the questions below :

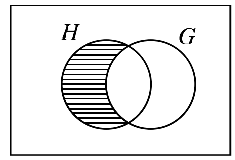
(Noted the x is used to denote the elements of the set).

(a) Show in Venn diagrams by shading.



(i) The number of students studying both Geography and History

(ii) The number of students studying only History.



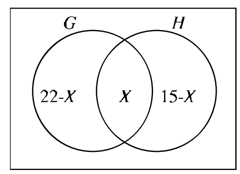
(b) in any class the number of students is 32, the number of students who study Geography is 22 and the number of students who study History is 15. Show in Venn diagram the number of students who study both Geography and History.

Solution : $(a)\,(i)$ $x \in H$ and $x \in G$

i.e., $x \in H \cap G$

(ii) $x \in H$ and $x \notin G$

i.e., $xH \setminus G$



(b) Let H be the set of student studying History G be the students studying Geograpy Then, the set of $H \cap G$ of students who study both Geography and History.

Let, $n(G \cap H) = x$, say.

Since everyone has studied at least one subject $H \cup G = \cup$

$n(H \cup G) = N(\cup)$

i.e., $(22 - x) + x + (15 - x) = 32$

$\Rightarrow$ $37 - x = 32$

$\therefore$ $x = 5$.

So, 5 students study both Geography and History.

Example 16. Each of the 35 girls in a class has to participate in at least one activity of: running, swimming, dancing. Among them 15 participate in running, 4 participate in swimming and dancing, 7 participate in jumping and swimming but not in dancing. Of them 20 do not like running, x like swiming and dancing $2x$ like only dancing, while 2 like swimming.

(a) Show the data in Venn diagram.

(b) Find x .

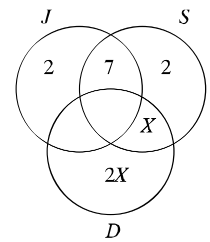
(c) Express the set [the girls who like running and dancing but not swimming]

(d) Find the number of girls who like running and dancing but not swimming.

Solution : (a) Suppose, set J = those who like running

S = those who like swimming

D = those who like dancing



(b) J' = j the girls who do not like running

$n(J') = 20$

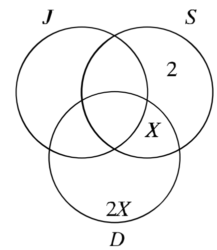
or, $2x + x + 2 = 20$

or, $3x = 18$

$x = 6$

(c) {The girls like running and dancing but not swimming}

$J \cap D \cap S'$.



(d) Let $n(J \cap D \cap S') = y$.

Given that $n(J) = 15$.

$y + 4 + 7 + 2 = 15$

$y = 2$

So, 2 girls like running and dancing but not swimming.

Example 17. Out of 24 students of 18 like to play basketball, 12 like to play volleyball. Given that U = {the set of the students}, B = {be the set of students who like to play basketball}, V = {the set of students who like to play vollyball}. Suppose $n(B \cap V) = x$ and explain the data below in Venn diagram:

(a) Describe the set $B \cup V$ and express $n(B \cup V)$ in terms of x.

(b) Find the minimum possible value of x.

(c) Find the maximum possible value of x.

Solution :

(a) $B \cup V$ is the set of the students who like to play basketball or vollyball.

$n(B \cup V) = (18 - x) + x + (12 - x) = 30 - x$

$\therefore$ $n(B \cup V) = (18 - x) + x + (12 - x) = 30 - x$

(b) $n(B \cup V)$ is the smallest, when $B \cup V = U$, so $n(B \cup V) = n(U)$ = 30 – x = 24

or, x = 6.

$\therefore$ the smallest possible value of x = 6

(c) $n(B \cap V)$ is the largest when $V \subseteq B = U$, so $n(B \cap V) = n(V) = x = 12$.

$\therefore$ the largest possible value of x = 12.

Activity :

1. Out of 30 students of a class, 20 students like football and 15 like cricket. Every student likes at least one of two the games. How many students like both the games ?

2. Among a certain group of persons, 50 can speak Bangla, 20 can speak English, and 10 can speak both Bangla and English. How many of these persons can speak at least one of the two languages ?
3. Out of 100 students of the Institute of Modern Languages of the University of Dhaka, 42 have taken French, 30 have taken German, 28 have taken Spanish, 10 have taken French and Spanish, 8 have taken German and Spanish, 5 have taken German and French, while 3 students have taken all three Languages..
(i) How many students have taken none of the three languages ?
(ii) How many students have taken just one of the three languages ?
(iii) How many students have taken precisely two of the three languages ?
4. Out of 50 students of Class Nine of a school, 29 have taken Civics, 24 have taken Geography, 11 have taken both Civics and Geography. How many students have taken neither Civics or Geography ?

Exercise 1·1

1. (i) A set of $2n$ elements has in all, 4^n subsets.
(ii) $Q = \left\{ \frac{p}{q} : p, q \in Z, q \neq 0 \right\}$ is the set of rational numbers.
(iii) $a, b \in R$; $]a, b[= \{x : x \in R$ and $a < x < b\}$

Which combination of these statements is correct ?
a. i and ii b. ii and iii c. i and iii d. i, ii and iii

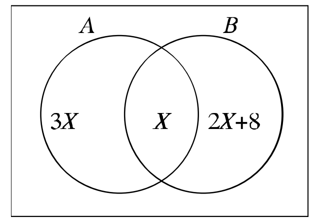
Answer the question (2-4) following the information below :
For every $n \in N$, $A_n = \{n, 2n. 3n. ....\}$.

2. Which one of the following is the value of $A_1 \cap A_2$?
a. A_1 b. A_2 c. A_3 d. A_4
3. Which one of the following denotes the value of $A_3 \cap A_6$?
a. A_2 b. A_3 c. A_4 d. A_6
4. Which one of the following is to be written down instead of $A_2 \cap B_3$?
a. A_3 b. A_4 c. A_5 d. A_6
5. Given that
$U = \{x : 3 \leq x \leq 20, n \in Z\}$,
$A = \{x : x$ is an odd
number } and $B = \{x : x$ is a prime number},
list the elements of the following sets :

(a) A and B

(b) Let $C = \{x : x \in A \text{ and } x \in B\}$ and $D = \{x : x \in A \text{ or } x \in B\}$.
describe the sets C and D.

6. The elements of the sets A and B have been shown in the Venn diagram. If $n(A) = n(B)$, find



(a) the value of x

(b) $n(A \cup B)$ and $n(A \cap B')$

7. The elements of each of the sets A and B have been shown in the Venn diagram Find $n(A' \cap B')$.
(a) the value of x (b) $n(A)$ and $n(B)$.

8. If $U = \{x : x$ is a positive integer$\}$, $A = \{x : x \geq 5\}$ and $B = \{x : x < 12\}$; find the value of $n(A \cap B)$ and $n(A')$

9. Let $U = \{x : x$ is an even integer$\}$, $A = \{x : 3x \geq 25\}$ and $B = \{x : 5x < 12\}$; find $n(A \cap B)$ and $n(A' \cap B')$.

10. Show that (a) $A \setminus A = \Phi$ (b) $A \setminus (A \setminus A) = A$.

11. Show that, $A \times (B \cup C) = (A \times B) \cup (A \times C)$

12. If $A \subset B$ and $C \subset D$, show that $(A \times C) \subset (B \times D)$

13. Show that the sets $A = \{1,2,3,........,n\}$ and $B = \{1,2,2^2,........2^{n-1}\}$ are equivalent.

14. Show that the set $S = \{1,4,9,25,36,............\}$ of square of natural numbers, is an infinite set.

15. Suppose, $A \cap B = \phi$ and $n(A) = p, n(B) = q$; prove that, $n(A \cup B) = p + q$.

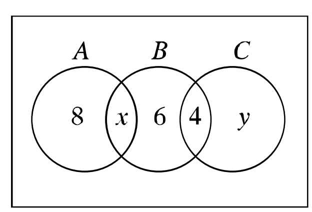
16. For finite sets A, B, C, prove that,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

17. If the are the sub set of sets $A = \{a, b, x\}$ and $B = \{c, y\}$, $U = \{a, b, c, x, y, z\}$ verify that $(a)(i) A \subset B'$, $(ii)\ A \cup B' = B'$, $(iii)\ A' \cap B = B$
(b) Find : $(A \cap B) \cup (A \cap B')$

18. Out of 30 students of a class, 19 have taken Economics, 17 have taken Geography, 11 have taken Civics, 12 have taken Economics and Geography, 4 have taken Civics and Geography, 7 have taken Economics and Civics, while 5 have taken all three subjects. How many students have taken none of the three subjects ?

19. The elements of the universal set U and the subsets A, B, C have been presented in the Venn diagram.

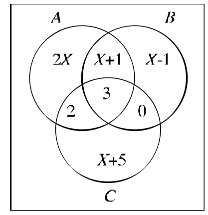


(a) If $n(A \cap B) = n(B \cap C)$, find the value of x.

(b) If $n(B \cap C') = n(A' \cap C)$, find the value of y.

(c) Find the value of $n(U)$.

20. The elements of the sets *A*, *B*, *C* have been given in the Venn diagram,
$U = A \cup B \cup C$.
(a) If $n(U)$ = 50, find the value of x .
(b) Find the values of $n(B \cap C')$ and $n(A' \cap B)$.
(c) Find the value of $n(A \cap B \cap C')$.



21. The three sets A, B, C have been given such that, $A \cap B = \Phi, A \cap C = \Phi$ and $C \subset B.$ Explain the sets by drawing Venn diagram.

22. Given that $A = \{x : 2 < x \le 5\}, x \in R$ and $B = \{x : 1 \le x < 3, x \in R\}$, $C = \{2,4,5\}$ Express the following sets in such set notation :
(a) $A \cap B$ (b) $A' \cap B'$ and (c) $A' \cap B$

23. Given that $U = (x : x < 10, x \in R\}$, $A = \{x : 1 < x \le 4\}$ and $B = \{x : 3 \le x < 6\}$. Express the following sets in such set notation :
(a) $A \cap B$ (b) $A' \cap B$ (c) $A \cap B'$ and (d) $A' \cap B'$

24. The sets A and B have been given below. Find in each case $A \cup B$ and verify that $A \subset (A \cup B)$ and $B \subset (A \cup B)$
i. $A = \{-2. -1, 0, 1, 2\}$ and $B = \{-3, 0, 3\}$
ii. $A = \{x : x \in N, x < 10$ and x is a multiple of 2} and
$B = \{x : x \in N, x < 10$ and x is a multiple of 3}.

25. Find $A \cap B$ by using the sets below and verify that $(A \cap B) \subset A$ and $(A \cap B) \subset B$. where (i) *A* = {0, 1, 2, 3,5}, *B* = {–1, 0, 2} (ii) $A = \{a, b, c, d\}$, $B = \{b, x, c, y\}$

26. Among the girls of Anwara College, a survey was conducted about their reading habits of the magazines of the Bichitra, the Sandhani and the Purbani. It was found that 60% of the girls read the Bichitra, 50% read the Sandhani, 50% read the Pubani, 30% read the Bichitra and the Sandhani, 30% read the Bichitra and the Purbani, 20% read the Sandhani and the Purbani, while 10% read all three magazines.
(i) What percentage of the girls do not read any of the three magazines ?
(ii) What percentage of the girls read just two of the above magazines ?

27. $A = \{x : x \in R$ and $x^2 - (a+b)x + ab = 0\}, B = \{1,2\}$ and $C = \{2,4,5\}$.
a. Find the elements of the set A .
b. Show that $P(B \cap C) = P(B) \cap P(C)$
c. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

28. Out of 100 students of a class, 42 students play football, 46 play cricket and 39 play hockey. Among them 13 play football and cricket, 14 play cricket and hocky and 12 play football and hockey. Besides 7 students play none of these games.

a. Show the set of students who play the above three games or none of the games in Venn diagrram.

b. Find how many students play all three games.

c. How many students play just one game ? How many play just two of the games ?

Relations and Functions

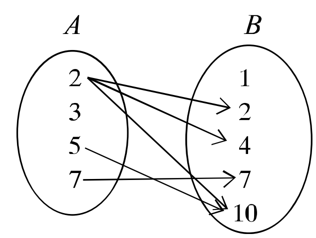
3 is greater than 2; the square of 3 is 9. These are the examples of relations. In the first example, the word 'greater than' has been defined which is included in the set *R* of all real numbers. In the second example the relation 'the square of' is also included in the set *R* of real numbers.

It is noted that all real numbers are included in the set R but all are not related. For example, 3, a prime number is not a relation because 3 is not related to another number which is shown in the example 1 by the Venn diagram.

Example 1. Suppose $A = \{2,3,5,7\}$ and

$B = \{1,2,4,7,10\}$

those element of *B* which are divisible by the elements of *A* are show below by drawing the Venn diagram.



We now form the set *D* = {(2, 2), (2, 4), (2, 10), (5, 10), (7, 7)}. of all ordered pairs which describe the relation of disivility.

In the set D the first component of the included ordered pairs belongs to A, the second components belongs to B and is divisible by the first component. So $D \subset A \times B$ and $D = \{(x, y) : x \in A, y \in B$ and y is divisible by $x\}$. Here the set *D* is relation from the set *A* to the set *B*.

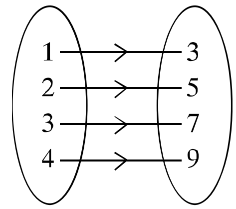
Example 2. Considering the set of ordered pairs of real numbers $L = \{(x, y) : x \in R, y \in R$ and $x < y)\}$ that $a < b$ is for the two real number *a*, *b*, if and only if it is $(a, b) \in L$.

Therefore, the set *L* describes the relation, less or greater.

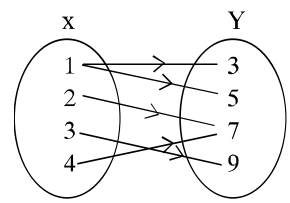
Definition : If A and B are sets, any non-empty subset of $A \times B$, is called relation from A to B.

Definiton : If A is any set, any non-empty subset of $A \times A$ is a relation from A to A.

Remark : Every relation is a set of one or more ordered pairs. In the Venn diagram, x the elements of x = {1, 2, 3, 4} and the elements of the set y = {3, 5, 7, 9} are associated with the arrow sign :



Such association between the sets X and Y is a relation from X to Y. In the Venn diagram, the element 1 of X related with the element 3 of Y is expressed by 1 $\rightarrow$ 3. We can say initial element 1 and through element 3 or 1 mapping into 3.



Similarly, 2 $\rightarrow$ 5, 3 $\rightarrow$ 7 and 4 $\rightarrow$ 9.

Every elements x of X are related with only one elements y of Y.

So, the relaion is called the function or mapping.

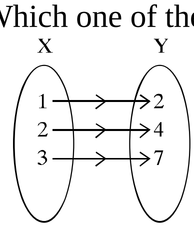
Consider another relation in the Venn diagram.

In the figure, it is seen, $1 \rightarrow 3, 1 \rightarrow 5, 2 \rightarrow 7, 3 \rightarrow 9$ and $4 \rightarrow 7$.

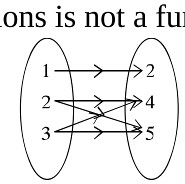
The two elements, 3 and 5 of Y are associated with the element 1 of X. Is the relation a function ? Give reasons.

Example 3. Which one of the following relations is not a function ? Give reasons.

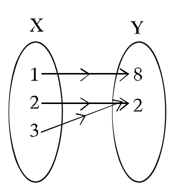
(a)



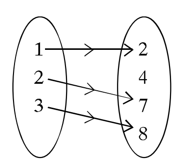
(b)



(c)



(d)



Solution : The relation in (a) and (d)

(b) is not a function, because 3$\rightarrow$ 4 and 3 $\rightarrow$ 5.

(c) depicts a function. But the relation in because 1 $\rightarrow$ 8, 2 $\rightarrow$ 2, 3 $\rightarrow$ 2..

Function is, generally denoted by the small letters f. g, h. etc of English Alphabet.

From $X = \{1, 2, 3, 4\}$ to Y = {3, 5, 7, 9} we write to denote function f : 1 $\rightarrow$ 3, f : 2 $\rightarrow$ 5, f : 3 $\rightarrow$ 7 and f : 4 $\rightarrow$ 9

Here 3 is the image of 1.

Similarly 5, 7, 9 are the image of 2, 3, 4 respectively.

Further we can express the relation of any element x of X and any element y of Y by $y = 2x + 1$. Therefore we can express functon in the following rule

$f : x \rightarrow y$ where $y = 2x + 1$

or, $f : x \rightarrow 2x + 1$

There we can write $f(x) = 2x + 1$

Then $f(1) = 3$ is the image of 1 and $f(x)$ is the image of x.

From the discussed function the 1st set $X = \{1, 2, 3, 4\}$ is the domain and the set of related that is set of related elements of 2nd set Y is the range of the function.

In other words, the domain of the function $y = f(x)$ is such a set of x where the value of the $f(x)$ for all the x is possible to determine. And all the values of $f(x)$ are to be detrmined for the domain of x. This collection is called the range.

Example 4. Find the image of the function $f : x \rightarrow 2x^2 + 1$: where the domain is $X = \{1, 2, 3\}$.

Solution : Here $f(x) = 2x^2 + 1$; so

The image of 1, 2, 3 : $f(1) = 2(1)^2 + 1 = 3, f(2) = 2(2)^2 + 1 = 9$ and $f(3) = 2(3)^2 + 1 = 19$

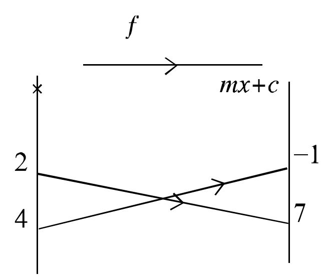
$\therefore$ Image set $R = \{3, 9, 19\}$.

Example 5 : Find from the part of the fuction $f : x \rightarrow mx + c$ in the adjoining picture.

(a) the values of m and c.

(b) the image of 5 under f.

(c) If the image 3, the number of element.



Solution : (a) $f(x) = mx + c$

we get $f : 2 \rightarrow 7$ and $\Rightarrow f(2) = 7$

$2m + c = 7$ .... (1)

i.e., $f : 4 - 1 \Rightarrow f(4) = -1$

i.e., $f(4) : 4m + c \Rightarrow 4m + c$ .... (2)

From (i) and (ii) we get, $m = -4$ and $c = 15$.

(b) image of under f is $f(5) = -4 \times 5 + 15 = -5$

(c) Suppose determining x the number of elements whose image 3, then, $f(x) = 3$

$\Rightarrow -4x + 15 = 3 \Rightarrow x = 3$

Example 6. Is the relation $F = \{(-2,4), (-1,1), (0,0), (1,1), (2,4)\}$ a function ? Find its domain and range.

Solution : The relation A = {(–3, 4), (–1, 1),(0, 0), (1, 1), (2, 4)} is a function, because the first component of its ordered pairs is different.
In the described function F (-2) = 4, F(-1) = 1, F(0) = 0, F(1) = 1, F(2) = 4. ;
$\therefore$ domain $A = \{-2, -1, 0, 1, 2\}$. and range B = {0, 1, 4}.
Observing the different image of the elements of A, it is found that $F(x) = x^2$ for $x \in A$. This function can be written $F : A \rightarrow B, F(x) = x^2$.

Remark : A function F is determined if its domain is defined and for every element x of the domain its image under F is uniquely specified. Sometimes the domain is kept implied ; in such cases we take, the sub set R where F(x) remains fixed in R for its each element x.

Example 7. Find the domain of the function $F(x) = \sqrt{1-x}$.
Find $F(-3), F(0), F\left(\frac{1}{2}\right), F(1), F(2)$ which ever is defined.

Solution : $F(x) =< \sqrt{1-x} \in R$ if and only if $1 - x \geq 0$ or, $1 \geq x$ i.e., $x \leq 1$
Therefore, Dom. $F = \{x \in R : x \leq 1\}$
Here, $F(-3) = \sqrt{1-(-3)} = 4 = 2$

$$F(o) = \sqrt{1-0} = \sqrt{1} = 1$$

$$F\left(\frac{1}{2}\right) = \sqrt{1-\frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

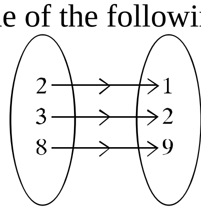
$$F(1) = \sqrt{1-1} = 0$$

$F(2)$ is undefined, because $2 \notin DomF$.

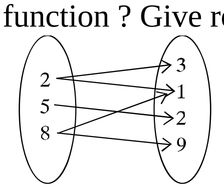
Activity :

1. Which one of the following relations is not function ? Give reasons.

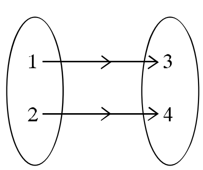
(a)



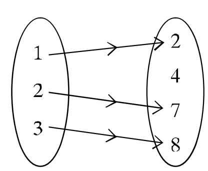
(b)



(c)



(d)

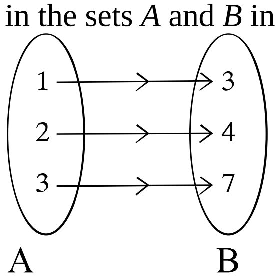


2. A function is defined by $f : x \rightarrow 4x + 2$ with domain $D = \{-1, 3, 5\}$; then find its image set :

3.	Describe the given relation S in roaster method and find which of these are function. Find Dom S and range S, where $A=\{-2, -1, 0, 1, 2\}$ (a) $S=\{(x,y): x\in A, y\in A,$ and $x+y=1\}$ (b) $S=\{(x,y): x\in A, y\in A,$ and $x-y=1\}$ (c) $S=\{(x,y): x\in A, y\in A,$ and $y=x^2\}$ (d) $S=\{(x,y): x\in A, y\in A,$ and $y^2=x\}$
4.	For the function defined by $f(x)=2x-1$ (a) Find $F(-2), F(0)$ and $F(2)$ (b) Find $F\left(\frac{a+1}{2}\right)$ where $a\in R$ (c) Find x, if $F(x)=5$ (d) Find x, if $F(x)=y$, where $y\in R$.

One One Function

Observe in the sets A and B in the Venn diagram :



The images of different elements under the function in the Venn diagram are always different.

Definition : A function is called one-one if distinct images in the domain always have distinc imanges in the condeomain under that function.

From the definition, it is found that a function, $f: A\to B$ is called one-one function if it is $f(x_1)=f(x_2)\Rightarrow x_1=x_2$ where $x_1, x_2\in A$.

Example 8. Show that $f(x)=3x+5, x\in R$ is an one-one function.

Verify : Given that $f(x)=3x+5$.

Assume that $a,b\in R$, then $f(a)=3a+4$ and $f(b)=3b+5$

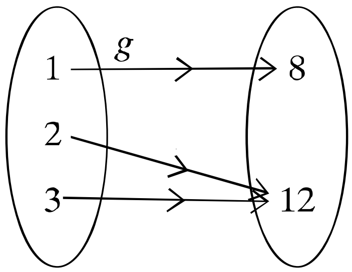
Then $f(a)=3a+5$ and $f(b)=3b+5$. So, $f(a)=f(b)$ implies. Now $f(a)=f(b)\Rightarrow$

$3a+5=3b+5\Rightarrow a=b$

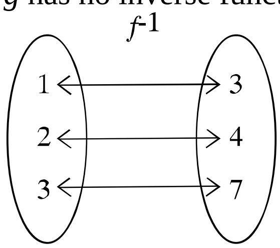
Therefore f is one-one function.

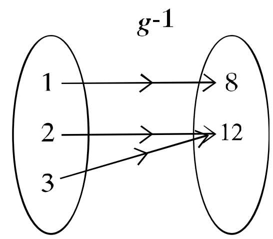
Image of two different elements in the domain are the same i.e., $g(2)=12, g(3)=12$

So the function g (figures) is not one-one,



If we reverse the direction of the arrow sign, we see that the first function is one-one. This function is called the inverse function. But the reverse maping under *g* is not one-one. So *g* has no inverse function.





Here the relaton between one-one function and its inverse are shown below :

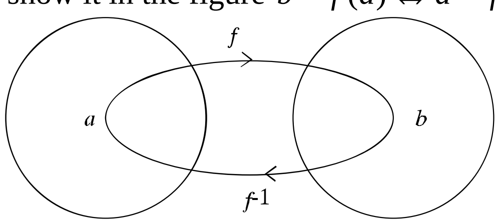
under f	under f^{-1}
$f(1)=2 \Leftrightarrow$	$f^{-1}(2)=1$
$f(2)=4 \Leftrightarrow$	$f^{-1}(4)=2$
$f(3)=7 \Leftrightarrow$	$f^{-1}(7)=3$
$f(a)=b \Leftrightarrow$	$f^{-1}(3)=a$

f is a one-one function if,

(i) the inverse function of f is f^{-1}

(ii) image b is under $f(a) \Leftrightarrow$ image a is $b = f(b)$ i.e., $b = f(a) \Leftrightarrow a = f^{-1}(b)$

we can show it in the figure $b = f(a) \Leftrightarrow a = f^{-1}(b)$



Example 9. Show that the function $F : R \to R, F(x) = x^2$ is not one-one.

Solution : Here domain. $F = R$ Taking $x_1 = -1, x_2 = 1, x_1 \in$ dom $F, x_2 \in$ dom F $x_1 \neq x_2$ But $F(x_1) = F(-1) = (-1)^2 = 1$, $F(x_2) = F(1) = 1^2 = 1$.

i.e., $F(x_1) = F(x_2)$,

$\therefore$ F is not one-one.

Note : Inverse relation of a function even not a function.

Example 10. Find (a) $f(5)$ (b) $F^{-1}(2)$ for the function defined by $F(x)=\dfrac{x}{x-2}, x\neq 2$.

Solution : (a) $f(x)=\dfrac{x}{x-2}, x\neq 2$

$$f(5)=\frac{5}{5-2}=\frac{5}{3}=1\frac{2}{3}$$

(b) Let, $a=f^{-1}$ then $f(a)=2$

$$\frac{a}{a-2}=2\Rightarrow a=2a-4\Rightarrow a=4$$

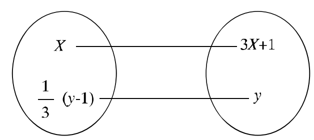
$\therefore$ $f^{-1}(2)=4$

Example 11. $f(x)=3x+1, 0\leq x\leq 2$

(a) Draw the graph of f and find its range.

(b) Show that f is one-one function.

(c) Determine f^{-1} and draw the graph of f^{-1}.



Solution : (a) From $f(x)=3x+1, 0\leq x\leq 2$

We have the vertices (0, 1) and (2, 7)

$\therefore$ range $f: R=\{y: 1\leq y\leq 7\}$

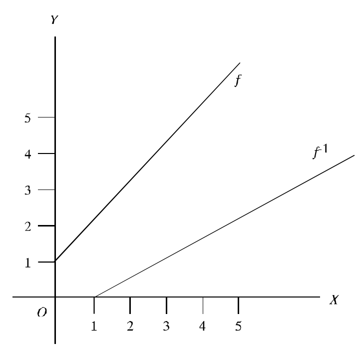
(b) Since the image y of only $x\in R$ is shown for every $y\in R$.

So f is a one-one.

(c) Let $y=f(x)$ is the image of x

Then $y=3x+1$

$\Rightarrow x=\dfrac{1}{3}(y-1)$



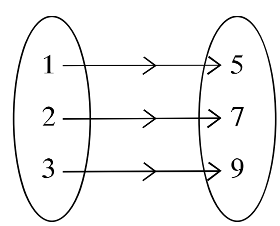
So the inverse function $f^{-1}: y\rightarrow x$

where $x=\dfrac{1}{3}(y-1)$.

or, $f^{-1}: y\rightarrow \dfrac{1}{3}(y-1)$ shown in the figure

Substituting x in place of y $f^{-1}: x\rightarrow \dfrac{1}{3}(x-1)$

Draw line of f^{-1} where $y=\dfrac{1}{3}(x-1), 1\leq x\leq 7$ is shwon.



Onto function :

In Venn diagram we consider the

function F under the sets $A = \{1, 2, 3\}$ and $B = \{5, 7, 9\}$
Here we can write $f(1) = 5 = 2.1 + 3$
Similarly $f(2) = 7 = 2.2 + 3$
$f(3) = 9 = 2.3 + 3$
Generally we can write $y = f(x) = 2x + b$
Therefore, it is said $f(a) = b$.

Definiton : A function $f : A \to B$ will be called the onto function if $f(A) = B$ or the function $f : A \to B$ is called onto function if for every $b \in B$, there exists $x \in A$ such that $f(a) = b$. The discussed function $f : A = \{1,2,3\} \to B = \{5,7,9\}$ is defined by $y = f(x) = 2x + 3.$
So f is an onto function.

Inverse function :

Suppose $f : A \to B$ is an one-one and onto function then the function $f^{-1} : B \to A$ for every $b \in B$. There exists an unique $f^{-1}(b) \in A$, then f^{-1} is called the inverse of f.

Suppose $f : A \to B$ and $g : A \to B$ are both one-one and onto fucntion, so g is called the inverse of f if $f(g(x)) = g(f(x)) = x$ where $f(x) \in B$ and $g(x) \in A$. Here $g = f^{-1}$.

Example 12. If the functions $f : R \to R$ and $g : R \to R$ are defined by
$f(x) = x + 5$ and $g(x) = x - 5$, show that the inverse of f is g.

Solution : Given that $f(x) = x + 5$ .... (i)
and $g(x) = x - 5$......... (ii)
Now g $f(g(x)) = f(x - 5)$ [by (ii)]
$= (x - 5) + 5$ [by (ii)]
$= x$
and $g(f(x)) = f(x) - 5$ [by (ii)]
$= (x + 5) - 5$ [by (i)]
$= x$
Since $f(g(x)) = g(f(x)) = x$
Therefore, the inverse function of f is g.

Example 13. If the function $f : R \to R$ is defined by $f(x) = x^3 - 5$, find $y = f^{-1}(x)$.

Solution : By definition, $f(f^{-1}(x)) = x$
$f(y) = x$........(i) [Since $f^{-1}(x) = y$]
Given that not value $f(x) = x^3 - 5$

$\Rightarrow f(y) = y^3 - 5$ [by (i)]

$\Rightarrow x = y^3 - 5 \Rightarrow x + 5 = y^3$

$\Rightarrow y = (x+5)^3$

$\therefore f^{-1}(x) = (x+5)^{\frac{1}{3}}$

Activity :

1. For each one-one function below find the related f^{-1} :

   (a) $y = (x+5)^{\frac{1}{3}}$ (b) $f(x) = \dfrac{3}{x-1}, x \neq 1$

   (c) $f(x) = \dfrac{2x}{x-2}, x \neq 2$ (d) $f : x \to \dfrac{2x+3}{2x-1}, x \neq \dfrac{1}{2}$

2. Given $f(x) = \dfrac{4x-9}{x-2}, x \neq 2$ of the function

   (a) Find $f^{-1}(-1)$ and $f^{-1}(1)$

   (b) Find the value of x such that $4f^{-1}(x) = x$

3. Given $f(x) = \dfrac{2x+2}{x-1}, x \neq 1$ of the function.

   (a) Find $f^{-1}(3)$

   (b) If $f^{-1}(p) = kp$, express k in terms of p.

4. Ascertain whether each relation F below is a function. If F is a function, find tits domain and range, Also ascertain whether it is one-one :

   (a) $F\{(x, y) \in R^2 : y = x\}$

   (b) $F\{(x, y) \in R^2 : y = x^2\}$

   (c) $F\{(x, y) \in R^2 : y^2 = x\}$

   (d) $F\{(x, y) \in R^2 : y = \sqrt{x}\}$

5. (a) If $f : \{-2 -1, 0, 1, 2\} \to \{-8, -1, 0, 1, 8\}$ is defined by $f(x) = x^3$, show that f is one-one and onto.

   (b) $f : \{1, 2, 3, 4\} \to R$ is a function which is defined by $f(x) = 2x + 1$. Show that f is a one-one function but not an onto function.

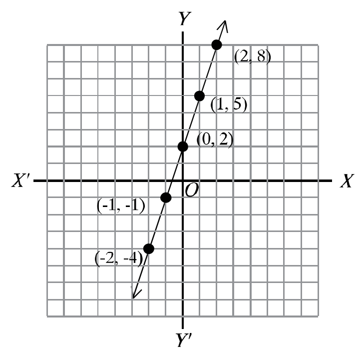
Graphs of Relations and Functions.

Graphs are useful geometrical representations of functions. To draw the graph of $y = f(x)$ we choose a point O as origin and draw through O a pair of perpendicular bisectors XOX' and YOY'. XOX' is called the x-axis and YOY' is called the y-axis.

For drawing the graph of $y = f(x)$, we choose an interval $a \le x \le b$ and tabulate the pairs of values of independent variable *x* and dependent variable *y*. These the limited points of the tabulation are placed on the plane xy. Joining the obtained points with the straight line or the curved line, we get the graph of function y = f(x). In secondary Algebra, the inital conceptions about graphs have been imparted. Here, the linear function, the quadratic function and the construction of the graphs of the circle have been discussed.

Linear Functions

The general form of linear function is $f(x) = mx + b$ where *m* and *b* are real numbers. The graph of this function is a line. Where slope is *m* and intercept of y axis is *b*. Here, taking *m* = 3 and *b* = 2, we have f(*x*) = 3*x* +2.



From the described function, we can get following related values of *y* :

x and	–2	–1	0	1	2
y	–4	–1	2	5	8

$\therefore$ The graph of $f(x) = 3x + 2$ is shown below :

Quadratic Function

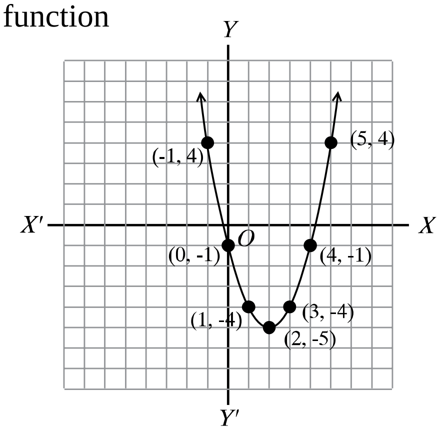
A quadratic function is a function which is described by the equation $y = ax^2 + bx + c$, where *a, b* and *c* are real numbers and $a \ne 0$.

We take the case $a = 1, b = -4, c = -1$

Then $y = ax^2 + bx + c$ can be written by $y = x^2 - 4x - 1$

We have the value of *x* and *y* from the described function

x	$x^2 - 4x - 1$	y
-1	$(-1)^2 - 4(-1) - 1$	4
0	$0^2 - 4(0) - 1$	-1
1	$1^2 - 4(1) - 1$	-4
2	$2^2 - 4(2) - 1$	-5
3	$3^2 - 4(3) - 1$	-4
4	$4^2 - 4(4) - 1$	-1
5	$5^2 - 4(5) - 1$	4



It is the required graph of a quadratic function.

Observe some general properties of the quadratic function.

(i) Paraboloid shape.

(ii) Symmetric point may be found about y axis or parallel of y axis.

(iii) The value of the function will be maximum or minimum at a certain point.

Graph of a Circle

Noted if p, q and r are constant and $r \neq 0$, in $R, S = \{(x, y) : (x - p)^2 + (y - q)^2 = r^2\}$

The graph of relation is a circle whose centre is the point (p, q) and radius is r (see Secondary Algebra).

In the graph paper ploting the point (p, q) as centre and r as radius, the drawn figure is the circle.

Coments : If the solution set of a relation is infinite, the assumpted system of ploting the sufficient repressentative point of solution in the graph paper is to join them so that the graph of the relation can be clear.

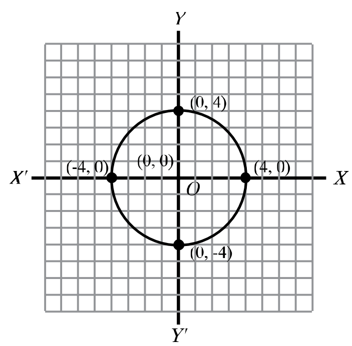
If the graph of relation is a circle the work becomes easier and beautiful for using compass, so we use the latter means..

Example 14. $S = \{(x, y) : x^2 + y^2 = 16\}$.

or, $x^2 + y^2 = 4^2$

Therefore, the graph of S is a circle with centre C (0, 0), and ratius x = 4.

Graph of S is shown below :



Activity :

1. Express the following functions in the standard form:

(a) $y - 2 = 3(x - 5)$ (b) $y - 2 = \frac{1}{2}(x + 3)$

(c) $y - (5) = -2(x + 1)$ (d) $y - 5 = \frac{4}{3}(x - 3)$

2.	Draw the graphs :	
	(a) $y = 3x - 1$	(b) $x + y = 3$
	(c) $x^2 + y^2 = 9$	(d) $y = \frac{1}{3}x + 1.$

Exercise 1·2

1. Which on is the domain of the relation $\{(2, 2), (4, 2), (2, 10), (7, 7)\}$?

   (a) $\{2, 4, 7\}$ (b) $\{2, 2, 10, 7\}$

   (c) $\{2, 2, 10, 7\}$ (d) $\{2, 4, 2, 5, 7\}$

2. Given $A = \{-2, -1, 0, 1, 2\}$ and $S = \{(x, y) : x \in A, y \in A$ and $y = x^2\}$ which of the following is a member of S ?

   (a) $(2, 4)$ (b) $(-2, 4)$

   (c) $(-1, 1)$ (d) $(1, -1)$

3. If $S = \{(1, 4), (2, 1), (3, 0), (4, 1), (5, 4)\}$:

   (i) The range of the relations S is $S = \{4, 1, 0, 4\}$

   (ii) The inverse relation of S is $S^{-1} = \{(4, 1), (1, 2), (0, 3), (1, 4), (4, 5)\}$

   (iii) S is a functon

Which combination of these statements is correct ?

   (a) i and ii (b) ii and iii (c) i and iii (d) i, ii and iii

Answer the questions 4 - 6 considering the following information :

4. If $F(x) = \sqrt{x - 1}$, $F(10) =$ what ?

   (a) 9 (b) 3 (c) -3 (d) $\sqrt{10}$

5. If $f(x) = 5$, what is the value of x ?

   (a) 5 (b) 24 (c) 25 (d) 26

6. Which one of the following is the domain of the funciton ?

   (a) Dom. $F = \{x \in R : x \neq 1\}$

   (b) Dom. $F = \{x \in R : x \geq 1\}$

(c) Dom. $F = \{x \in R : x \leq 1\}$

(d) Dom. $F = \{x \in R : x > 1\}$

7. (a) Find the domain and range of the relation and find the inverse relation.

(b) Ascertain whether S or S^{-1} is a function.

(c) Are the functions among these relations one-one ?

(i) $S = \{(1, 5), (2, 10), (3, 15), (4, 20)\}$

(ii) $S = \{(-3, 8), (-2, 3), (-1, 0), (0, -1), (1, 0), (2, 3), (3, 8)\}$

(iii) $S = \left\{\left(\frac{1}{2}, 0\right), (1, 1), (1, -1), \left(\frac{5}{2}, 2\right), \left(\frac{5}{2}, -2\right)\right\}$

(iv) $S = \{(-3, -3), (-1, -1), (0, 0), (1, 1), (3, 3)\}$

(v) $S = \{(2,1), (2,2), (2,3)\}$

8. For the described function by $F(x) = \sqrt{x-1}$:

(a) Find $F(1)$, $F(5)$, and $F(10)$ (b) Find $F(a^2 + 1)$ where $a \in R$

(c) If $F(x) = 5$, find x (d) If $F(x) = y$, find x where $y \geq 0$.

9. For the function $F : R \rightarrow R$ defined by $F(x) = x^2$ where

(a) Find dom F and range F (b) Show that F is not an one-one function.

10. (a) $f : R \rightarrow R$, is a function if defined by $f(x) = ax + b$, $a, b \in R$ show that is one-one and onto.

(b) If the function $f : [0, 1] \rightarrow [0, 1]$ defined by $F(x) = \sqrt{1 - x^2}$, show that f is one-one and onto.

11. (a) If the functions $f : R \rightarrow R$ and $g : R \rightarrow R$ are defined by $f(x) = x^3 + 5$ and $g(x) = (x-5)^{\frac{1}{3}}$, show that $g = f^{-1}$.

12. If R is the set of real numbers and the function $f : R \rightarrow R$ indefined by $f(x) = x^2 - x - 2$, find $f^{-1}[-2,0]$ and $f^{-1}(0)$

13. Draw the graph of the relation S and ascertain from the graph whether S is a function, where

(a) $S=\{(x,y):2x-y+5=0\}$

(b) $S=\{(x,y):x+y=1\}$

(c) $S=\{(x,y):3x+y=4\}$

(d) $S=\{(x,y):x=-2\}$

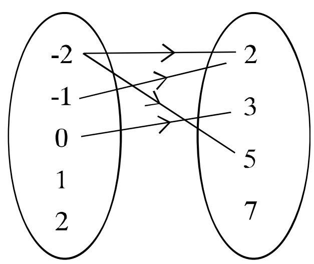
14. Draw the graph of the relation S and ascertain from the graph whether S is a function, where $x,y\in R$ and

(a) $S=\{(x,y):x^2+y^2=25\}$

(b) $S=\{(x,y):x^2+y^2=9\}$

15. $A=\{-2,-1,0,1,2\}$ and $B=\{2,3,5,7\}$, an association of certain elements of the set with elements of the set is depicted below :

(a) if the formed relation is D, express the value of D by the set of ordered pairs.



(b) Describing the relation $S=\{(x,y):x\in A, y\in A$ and $x=y^2\}$ in the roster method, find the domain and the range of S .

(c) Draw the graph of the relation described above, and ascertain from the graph whether the relation is a function.

16. Given $F(x)=2x-1$

(a) Find the value of $F(x+1)$ and $F\left(\frac{1}{2}\right)$.

(b) Ascertain whether the function F is one-one, when $x,y\in N$.

(c) If $F(x)=y$, find the three values of x when $x,y\in N$ and draw the graph of the equation $y=2x-1$.

Chapter Two

Algebraic Expressions

We are familiar with various types of algebraic expression. When one or more numbers and symbols representing numbers are combined meaningfully by any one sign or more than one +, –, ×, ÷ of power or rational sign, a new symbol representing numbers is crated. It is called an algebraic expression or in brief expression. For example, each of $2x, 2x+3ay,\ 6x+4y^2+a+\sqrt{z}$, is an algebraic expression.

After completing the chapter, the students will be able to :

- Explain the concept of a polynomial–
- Explain with example the concept of a plynomial with one variable
- Explain multiplication and division of polynomials
- Explain the remainder theorem and the factor theorem and apply to factorize a polynomial
- Explain homogeneous expressions, symmetric expressions and cyclic expressions
- Factorize homogeneous expression, symmetric expressions and cyclic expression
- Resolve rational fractions into partial fractions.

2.1 In this chapter by number we will mean real number. If A, B, C .... are expressions none of which is a sum or difference of more expressions, each of them is called a term of the expression $A+B+C+$.......... For example, in the expression , $5x+3y^2-2b+\sqrt{2}$ each of $5x, 3y^2, -2b, \sqrt{2}$, is a term.

In any discussion, a literal symbol representing a number may be a variable or a constant. If such a symbol denotes any unscheduled element of the number set consisting of more than one elements we call the symbol a variable and the set is called the domains. If the symbol denotes a definite number, it is called a constant.

In any discussion a variable can take any value from its domain, but the value of a constant remains fixed all through.

Polynomials

A polynomial is a special type of algebraic expression. In a polynomial there are one or more terms and each term in a product of a constant and a non-negative integral power of one or more variables.

Polynomials in One Variable :

Suppose x is a variable, Then (1) a , (2) $ax+b$ (3) ax^2+bx+c

(4) ax^3+bx^2+cx+d etc. like expressions are polynomials of the variable x; where a, b, c, d etc. are fixed numbers constants.

Generally, the terms of a polynomial of the variable x have the form Cx^p , where C (independent of x) is a fixed number (which may be zero) and p is a non-negative integar. If p is zero, the term becomes C, and if C is zero, the term is suppressed in the polynomial. C is called the coefficient of x^p in the term Cx^p and p is called the degree of the term.

The largest of the degrees of the terms appearing in a polynomial is called the degree of the polynomial. The term having the largest degree is called the leading term and the coefficient of the term having the largest degree is called the leading coefficient ; the term with degree 0, that is, the term independent of variale x, is called the constant term. For example, $2x^6 - 3x^5 - x^4 + 2x - 5$ is a polynomial of the variable x, its degree is 6, its leading term is $2x^6$ leading coefficient is 2 and constant is – 5. If $a \neq 0$, the aforesaid (1) the degree of the polynomial is 0, (2) the degree of the polynomial is 1, (3) the degree of the polynomial is 2 and (4) the degree of polynomial is 3. Any non zero constant $(a \neq 0)$ is a 0 degree polynomial of the variable (consider $(a = ax^o)$. The number 0 is considered a zero polynomial and the degree of a zero polynomial is considered underfined.

A polynomial of the variable x is usually arranged in descending order of the degrees x of its terms (the leading term appearing first and the constant term appearing last). This arrangement of a polynomial is called its standard form.

For convenience of using, polynomials of the variable x are denoted by symbols like $P(x), Q(x)$ etc. For example, $P(x) = 2x^2 + 7x + 5$

Such symbol $P(x), Q(x)$ indicates that the values of a polynomial depend on the values of x. If a specific number a is substituted for x in the polynomial, the value of the polynomial for that value of x is denoted by $P(a)$.

Example 1. If $P(x) = 3x^3 + 2x^2 - 7x + 8$, find $P(0), P(1), P(-2), P\left(\frac{1}{2}\right), P(2)$ and $P(a)$.

Solution : Replacing x successively by $0,\ 1,\ -2,\ \frac{1}{2},\ 2$ and a, we get

$$P(0) = 3(0)^3 + 2(0)^2 - 7(0) + 8 = 8$$

$$P(1) = 3(1)^3 + 2(1)^2 - 7(1) + 8 = 6$$

$$P(-2) = 3(-2)^3 + 2(-2)^2 - 7(-2) + 8 = 6$$

$$P\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^3 + 2\left(\frac{1}{2}\right)^2 - 7\left(\frac{1}{2}\right) + 8 = \frac{43}{8}$$

$$P(a) = 3a^3 + 2a^2 - 7a + 8.$$

Polynomials in Two Variables

$$2x + 3y - 1$$

$$x^2 - 4xy + y^2 - 5x + 7y + 1$$

$$8x^3 + y^3 + 10x^2y + 6xy^2 - 6x + 2$$

These are the polynomials of the variables x and y. The terms of such polynomials have the form $Cx^p x^q$ where C is a definite number (constant) and p, q are non-negative integers. In the term $Cx^p x^q$ the coefficient of $x^p x^q$ is C and the degree of the term is $p+q$, provided $C \neq 0$. In such polynomial, the mentioned largest of the degrees of the terms is called the degree of the polynomial. Such polynomials are denoted by $p(x, y)$. For exmple, the polynomial

$p(x, y) = 8x^3 + y^3 - 4x^2 + 7xy + 2y - 5$ has degree 3 and

$P(1, 0) = 8 - 4 - 5 = -1.$

Polynomials of Three Variables

The terms of a polynomial of the variable x, y and z have the form $Cx^p x^q z^r$ where C (constant) is the coefficient of the term and non-negative intergers are p, q, r; $(p+q+r)$ is the degree of this term and in the mentioned terms of the polynomial, the largest of the degrees is called the degree of the polynomial. Such polynomial is denoted by the symbol $P(x, y, z)$. For example, the polynomial $P(x, y, z)$ = $x^3 + y^3 + z^3 - 3xyz$ has degree 3, and $P(1, 1, -2) = 1 + 1 - 8 + 6 = 0$.

Note : Addition , Subtraction and Multiplication of two polynomial is always polynomial. But division of polynomial may or may not be polynomial.

Activity :

1. Ascertain which one of the following expressions is polynomial :

(a) $2x^3$	(b) $7 - 3a^2$	(c) $x^3 + x^{-2}$
(d) $\dfrac{a^2 + a}{a^3 - a}$	(e) $5x^2 - 2xy + 3y^2$	(f) $6a + 3b$
(g) $C^2 + \dfrac{2}{o} - 3$	(h) $3\sqrt{n-4}$	(i) $2x(x^2 + 3y)$
(j) $3x - (2y + 4z)$	(k) $\dfrac{6}{x} + 2y$	(l) $\dfrac{3}{4}x - 2y$

2. Ascertain the polynomials according to the variable :

(a) $x^2 + 10x + 5$	(b) $3a + 2b$	(c) $4xyz$
(d) $2m^2n - mn^2$	(e) $7a + b - 2$	(f) $6a^2b^2c^2$

3. Express each of the following polynomials (i) as a polynomial of the variable x in its standard form, and ascertain its degree, leading coefficient and constant term ; (ii) as a polynomial of the variable y in its standard form, and ascertain its as a polynomial in y degree, leading coefficient and constant term as a polynomial of the variable y.

(a) $3x^2 - y^2 + x - 3$ (b) $x^2 - x^6 + x^4 + 3$

(c) $5x^2y - 4x^4y^4 - 2$

(d) $x + 2x^2 + 3x^3 + 6$ (e) $3x^3y + 2xyz - x^4$

4. If $P(x) = 2x^2 + 3$, evaluate $P(5),\ P(6),\ P\left(\frac{1}{2}\right)$.

Division Algorithm :

If both $D(x)$ and $N(x)$ are the polynomials of the variable x and if degree of $D(x)$ $\leq$ (degree of $N(x)$), we can divide $N(x)$ by $D(x)$ in usual manner and we obtain a quotient $Q(x)$ and a remainder $R(x)$. where,

(1) Both $Q(x)$ and $R(x)$ are the polynomial of the variable x

(2) degree of $Q(x)$ = degree of $N(x)$ – degree of $D(x)$

(3) either $R(x) = 0$ or degree of $R(x) <$ degree of $D(x)$

(4) $N(x) = D(x)\ Q(x) + R(x)$ holds for all x.

Remarks : We can express the rule (4) as ;

Dividend = Devisor × quotient + Remainder.

Equality of Polynomials

(1) If $ax + b = px + q$ holds for all x, putting $x = 0$ and $x = 1$ we get repectively, $b = q$ and $a + b = p + q$, from whence it is found $a = p, b = q$.

(2) If $ax^2 + bx + c = px^2 + qx + r$ holds for x, putting $x = 0, x = 1$ and $x = -1$ we get respectively $c = r, a + b + c = p + q + r$ and $a - b + c = p - q + r$ from whence it is found $a = p, b = q, c = r$.

(3) In general, if $a_o x^n + a_1 n^{-1} + a_2 x^{n-2} + \dots\dots + a_{n-1}x + a_n$

$= p_o x^n + p_1 x^{n-1} + \dots\dots + p_{n-1}x + p_n$.

Holds for all x, so $a_o = p_o,\ a_1 = p_1, \dots\dots, a_{n-1} = p_{n-1}\ \ a_n = p_n$.

i.e., the two coefficients of x with the same power are equal in both sides of an equal sign.

Remarks : It is convenient to denote the coefficients of a polynomial of degree n by a_o (a sub - zero), a_1 (a sub - one), etc.

If the two polynomials $P(x), Q(x)$ for all x are equal, their equality is called the referred to as identity of the polynomials ; sometimes this is indicated by writing $P(x) \cong Q(x)$. In this case the two polynomials $P(x)$ and $Q(x)$ are the same. The sign $\cong$ is called the identity sign. Generally, the equality of the two algebraic expressions is called the identity, if the domain of anyone variable of the two expressions is the same and for the values included in the domain of the variables, the values of the two expressions are equal. For example, $x(y+z) = xy + xz$ is an identity.

2.2 Remainder and Factor Theorems

In this section we shall deal with polynomials of one variable only. First we consider the two examples.

Example 1. If $P(x) = x^2 - 5x + 6$, divide $P(x)$ by $(x-4)$ and show that the remainder is equal to $P(4)$.

Solution : We divide $P(x)$ by $x-4$;

$$
\begin{array}{l}
x-4)\, x^2 - 5x + 6\, (x-1 \\
\quad\; \underline{x^2 - 4x} \\
\qquad\quad -x + 6 \\
\qquad\quad \underline{-x + 4} \\
\qquad\qquad\; 2
\end{array}
$$

So, the remainder is 2. Now $P(4) = 4^2 - 5(4) + 6 = 2$

Therefore, the remainder is equal to $P(4)$.

Example 2. If $P(x) = ax^3 + bx + c$, divide $P(x)$ by $x - m$ and show that the remainder is equal to $P(m)$.

Solutin : We divide $P(x)$ by $x - m$:

$$
\begin{array}{l}
x-m)\, ax^3 + bx + c\, (ax^2 + amx + am^2 + b \\
\quad\; \underline{ax^3 - amx^2} \\
\qquad\quad amx^2 + bx + c \\
\qquad\quad \underline{amx^2 + am^2x} \\
\qquad\qquad (am^2 + b)x + c \\
\qquad\qquad \underline{(am^2 + b)x - (am^2 + b)m} \\
\qquad\qquad\qquad\qquad am^3 + bm + c
\end{array}
$$

So the remainder $= am^3 + bm - c$.

But $P(m) = am^3 + bm + c$, Therefore, the remainder is equal to $P(m)$.

These two examples suggest the following propositions.

Remainder Theorem

Proposition 1 : If $P(x)$ is a polynomial of positive degree and a is any definite number, the remainder of the division of $P(x)$ by $x - a$ will be $P(a)$.

Proof : If we divide $P(x)$ by $x - a$ the remainder is either 0 or a non-zero constant. Suppose, the remainder R and the quotient $Q(x)$, then by law of division, for all x.

$$P(x) = (x-a)Q(x) + R.............(i)$$

Putting $x = a$ in (i) we get, $P(a) = 0 \cdot \ Q(a) + R = R.$

Therefore, the remainder of $P(x)$ upon division by $x - a$ is $P(a)$.

Example 3. What will be the remainder if the polynomial $P(x) = x^3 - 8x^2 + 6x + 60$ is divided by $x + 2$?

Solution : Here $x - a = x + 2 \quad \therefore \ x + 2 = x - (-2) \Rightarrow a = -2$

Therefore, the required remainder is $P(-2) = (-2)^3 - 8(-2)^2 + 6(-2) + 60 = 8$

It is proved by following the proposition 1,

Proposition 2. If $P(x)$ is a polynomial of positive degree and $a \neq 0$, the remainder of the division of $P(x)$ by $ax + b$ will be $P\left(\frac{-b}{a}\right)$.

Example 4. What will be the remainder if the polynomial $P(x) = 36x^2 - 8x + 5$ is divided by $(2x - 1)$?

Solution : The required remainder is

$$P\left(\frac{1}{2}\right) = 36\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right) + 5 = 9 - 4 + 5 = 10$$

Example 5. Given that the remainder of $P(x) = 5x^3 + 6x^2 - ax + 6$ upon division by $x - 2$ is 6, find the value of a.

Solution : The remainder of $P(x)$ upon division by $x - 2$ is

$$P(2) = 5(2)^3 + 6(2)^2 - a(2) + 6$$
$$= 40 + 24 - 2a + 6$$
$$= 70 - 2a$$

By the given condition, $70 - 2a = 6$

or, $2a = 70 - 6 = 64 \Rightarrow a = 32$

Example 6. If $P(x) = x^3 + 5x^2 + 6x + 8$ yields the same remainder upon division by $x - a$ and $x - b$ where $a \neq b$, show that, $a^2 + b^2 + ab + 5a + 5b + 6 = 0$.

Solution : The remainder of $P(x)$ upon division by $x - a$ is $P(a) = a^3 + 5a^2 + 6a + 8$;

The remainder of $P(x)$ upon division by $x - b$ is $P(b) = b^3 + 5b^2 + 6b + 8$.

By the given condition, $a^3 + 5a^2 + 6a + 8 = b^3 + 5b^2 + 6b + 8$

or, $a^3 - b^3 + 5(a^2 - b^2) + 6(a - b) = 0$

or, $(a - b)(a^2 + b^2 + ab + 5a + 5b + 6) = 0$

$\therefore \ a^2 + b^2 + ab + 5a + 5b + 6) = 0$ (shown).

Because $(a-b) \neq 0$ i.e., $a \neq b$.

Factor Theorem

Propositon 3: If $P(x)$ is a polynomial of positive degree and if $P(a)=0$, so $x-a$ is a factor of $P(x)$.

Proof : The remainder of $P(x)$ upon division by $x-a$ is $P(a)$ [Remainder Theorem]

= 0 [given]

This means, the polynomial $P(x)$ is divisible by $x-a$.

$\therefore$ $x-a$ is a factor of the polynomial $P(x)$.

Converse of Factor Theorem

Proposition 4 : If $x-a$ is a factor of the polynomial $P(x)$, show that $P(a) = 0$.

Proof : Since $x-a$ is a factor of $P(x)$, there exists a polynomial $Q(x)$ that $P(x)=(x-a)Q(x)$.

Putting $x=a$ we get,

$$P(a)=(a-a)Q(a)=0.Q(a)=0 \text{ (proved).}$$

Example 7. Show that $x-1$ will be a factor of $P(x)=ax^3+bx^2+cx+d$, if and only if $a+b+c+d=o$.

Solution : Suppose, $a+b+c+d=0$

Then $P(1)=a+b+c+d=0$ [by conditions]

Therefore, $x-1$ is a factor of $P(x)$ [by factor theorem]

Now suppose $x-1$ is a factor of $P(x)$

So, by the converse of factor theorem we get, $P(1)=0$

i.e., $a+b+c+d=0$

Remark : In general, $x-1$ will be a factor of a polynomial of positive degree if and only if the sum of the coefficient of the polynomial is 0.

Example 8. Suppose, $P(x)=ax^3+bx^2+cx+d$ is a polynomial with integer coefficients, $a \neq 0$ and $d \neq 0$; suppose $x-r$ is a factor of $P(x)$ show that,

(a) If r is an integer, r will be factor of the constant term.

(b) If $r=\dfrac{p}{q}$ is a rational number in its reduced form, p will be a factor of d, and q will be a factor of a.

Proof : From factor theorem, we get, $P(r)=ar^3+br^2+cr+d=0$.

or, $(ar^2+br+c)r=-d$.

Since (ar^2+br+c), r and d are integers, this implies, r is a factor of d.

(b) From the factor theorem we get, $P(r) = ar^3 + br^3 + cr + d = 0$

$$= P\left(\frac{p}{q}\right) = a\left(\frac{p}{q}\right)^3 + b\left(\frac{p}{q}\right)^2 + c\left(\frac{p}{q}\right) + d = 0$$

$= ap^3 + bp^2q + cpq^2 + dq^3 = 0$ …………..(i)

From (i) we get, $(ap^2 + bpq + cq^2)p = -dq^3$ …………..(ii)

and $(bp^2 + cqp + dq^2)q = -ap^3$ …………..(iii)

Now, $ap^2 + bpq + cq^2, dp^2 + cpq + dq^2$, each of p, q, d, a is an integer.

From (ii) we get that p is a factor of dq^3; from (iii) we get that q is a factor of ap^3. But p and q, have no common factor except ± 1. Therefore, p is a factor of d, and q is a factor of a.

Remarks : From the above example we see that to determining the factor of integer coefficient of the polynomial $P(x)$ by factor theorem first we can test $P(r)$, then $P\left(\frac{r}{s}\right)$ where r is the factor with $r = \pm 1$ with the constant of polynomial and s is the factors with $s = \pm 1$ leading coefficient of the polynomial.

Example 9. Resolve the polynomial $P(x) = x^3 - 6x^2 + 11x - 6$ into factors.

Solution : All coefficiefficients of the given polynomial are integers and constant – 6, leading coefficient = 1. Now, if $P(x)$ as any factor of the form $x - r$, where r is an integer, r will be a factor of the constant term, which is -6. So possible values of r are ±1, ±2, ±3 and ±6.

Now we verify the values of $P(x)$ for these values of r.

$p(1) = 1 - 6 + 11 - 6 = 0 \quad \therefore\ x - 1$ is a factor of $p(x)$.

$p(-1) = -1 - 6 - 11 - 6 \neq 0 \quad \therefore\ x + 1$ is not a factor of $p(x)$.

$p(2) = 8 - 24 + 22 - 6 = 0 \quad \therefore\ x - 2$ is a factor of $p(x)$.

$p(-2) = -8 - 24 - 22 - 6 \neq 0 \quad \therefore\ x + 2$ is not a factor of $p(x)$.

$p(3) = 27 - 54 + 33 - 6 = 0 \quad \therefore\ x - 3$ is a factor of $p(x)$.

We have found the degree 3 of $P(x)$ and three factors of degree 1, if $P(x)$ has anyother factor, it will be the constant.

$\therefore\ P(x) = k(x-1)(x-2)(x-3)$ where k is constant.

Equating the coefficients of the highest power of x appearing on both sides, we get $k = 1$. Therefore, $P(x) = (x-1)(x-2)(x-3)$.

Remarks : To resolve into factor of the polynomial $P(x)$, first we determine the factor of type $x - r$ and then divide directly $P(x)$ by $x - r$ or rearrange the terms of

$P(x)$ in the form $P(x)=(x-r)Q(x)$. Where the degree of $Q(x)$ is less than the degree of $P(x)$ by 1. The proceed by determning $Q(x)$.

Example 10. Resolve into factors : $18x^3+15x^3-x-2$.

Solution : Let $P(x)=18x^3+15x^3-x-2$. Here the constant term is –2, and the set of its factors is $F_1=\{1, -1, 2, -2\}$ and the set of factors of the leading coefficient 18 is $F_2 = \{1, -1, 2, -2, 3, -3, 6, -6, 9, -9, 18, -18\}$ of $P(x)$.

Now consider $P(x)$ where $a=\dfrac{r}{s}$ and $r\in F_1, s\in F_2$

If $a=-1$, so $P(1)$ = 18 + 15 – 1 – 2 $\neq 0$

If $a=-1$, so $P(-1)$ = 18 + 15 – 1 – 2 $\neq 0$

If $a=\dfrac{-1}{2}$, so $P\left(-\dfrac{1}{2}\right)=-18\left(-\dfrac{1}{8}\right)^3+15\left(\dfrac{1}{4}\right)^2+\dfrac{1}{2}-2$

$$=\frac{-9}{4}+\frac{15}{4}+\frac{1}{2}-2$$

$$=\frac{17}{4}+\frac{-17}{4}=0$$

So, $x+\dfrac{1}{2}, \dfrac{1}{2}(2x+1)$, i.e., $2\left(x+\dfrac{1}{2}\right)=2x+1$ is a factor of $P(x)$.

Now, $18x^3+15x^2-x-2=18x^3+9x^2+6x^2+3x-4x-2$

$$= 9x^2(2x+1)+3x(2x+1)-2(2x+1)$$

$$= (2x+1)(9x^2+3x-2) \ ;$$

And $9x^2+3x-2=9x^2+6x-3x-2$

$$= 3x(3x+2)-1(3x+2)$$

$$=(3x+2)(3x-1)$$

$\therefore$ $P(x)=(2x+1)(3x+2)(3x-1)$ is the desired factorization.

Activity :

1. Find the remainder of $P(x)=2x^4-6x^3+5x-2$ upon division by each of the following polynomials.
   (i) $x-1$ (ii) $x-2$ (iii) $x+2$ (iv) $x+3$
   (v) $2x-1$ (vi) $2x+1$
2. Find the remainder by using the Remainder Theorem :
   (i) Dividend : $4x^3-7x+10$, Divisor : $x-2$
   (ii) Dividend : $5x^3-11x^2-3x+4$, Divisor : $x+1$

(iii) Dividend : $2y^3 - y^2 - y - 4$, Divisor : $y + 3$

(iv) Dividend : $2x^3 + x^2 - 18x + 10$, Divisor : $2x + 1$

3. Show that $(x-1)$ is a factor of $3x^3 - 4x^2 + 4x - 3$.
4. If $x+3$ is a factor of the polynomial $2x^3 + x^2 + ax - 9$, find the value of a.
5. Show that $x-3$ is a factor of the polynomial $x^3 - 4x^2 + 4x - 3$.
6. Find the remainder when by using the Remainder Theorem :
   $P(x) = 2x^3 - 5x^2 + 7x - 8$ is divided by $x - 2$.
7. Show that $x+1$ and $x-1$ are two common factors of the polynomial $4x^3 - 5x^2 + 5x - 2$.
8. Resolve into factors :
   (i) $x^3 + 2x^2 - 5x - 6$ (ii) $x^3 + 4x^2 + x - 6$
   (iii) $a^3 - a^2 - 10a - 8$ (iv) $x^4 + 3x^3 + 5x^3 + 8x + 5$

2.3 Homogeneous, Symetric and Cyclic Expressions

Homogeneous Polynomial : If each tern of any polynomial has same degree, it is called homogeneous polynormial. The expression $x^2 + 2xy + 5y^2$ is a homogeneous polynomial of the variable x and y With two degree (each term having degree 2). It is a special case of the homogeneous polynomial $ax^2 + 2hxy + by^2$ with having two degree of homogeneous expression of two variables $x, y,$ where a, h, b are definite numbers. Considering x, y, a, h and b the variables, $ax^2 + 2axy + by^2$ is homogeneous polynomial of degree 3.

$2x^2y + y^2z + 9z^2x - 5xyz$ is a homogeneous polynomial of the variables x, y, z, each term having degree 3.

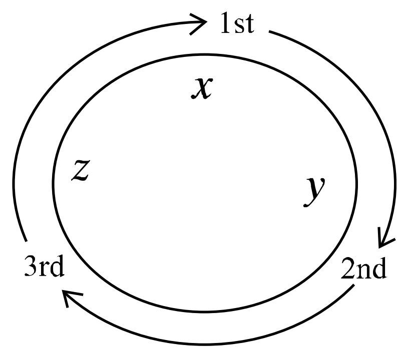
Symmetric Expression

A symmetric expression is an expression with more than one variable, which remains unchanged when any two of its variables are interchanged.

The expression $a+b+c$ is symmetric expression of the variables, a, b, c because the expression remains unchanged when any two of the variables are interchanged. Similarly $ab + bc + ca$ of the variables a,b,c and $x^2 + y^2 + z^2 + xy + yz + zx$ are symmetric expression of the variables x, y, z.

But $2x^2 + 5xy + 6y^2$ is not a symmetric expression of the variables x, y because interchanging x and y it becomes $2x^2 + 5xy + 6y^2$ which is different from former expression.

Cyclic Expression :



A cyclic expression is an expression with three variables, which remain unchanged when the first variable is replaced by the second, the second variable is replacd by the third and the third variable is replaced by the first variable.

If the replacing of the variables is done cycle-wise like the adjacent figure, it is called cyclically symmetic expression. $x^2+y^2+z^2+xy+yz+zx$ is a cyclic expression of the variables x, y, z because replacing y by x, z by y and x by z, the expression remains the same. Similarly the expression $z^2+x^2+y^2+zx+xy+yz$ is a cyclic expression of variable *x, y* and *z*.

The expression $x^2-y^2+z^2$ is not cyclic, because replacing y by x, z by y and x by z, the expression becomes $y^2-z^2+x^2$ which is different from the former expression.

Clearly, every symmetric expression in three variables is cyclic. But not every cyclic expression is symmetric. For example, the expression : $x^2(y-z)+y^2(z-x)+z^2(x-y)$ is cyclic but not symmetric. For interchanging x and y it becomes $x^2(y-z)+z^2(z-y)+z^2(y-x)$, which is different from the former expression.

Remarks : For convenience of description, the expression of variable x and y is denoted by $F(x, y)$ and that of variable *x, y* and *z* is denoted by $F(x, y, z)$.

Exmaple 1. Show that the expression $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$ is cyclic but not symmetric.

Taking $\left[F(x,y,z)=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right]$ do yourself.

Resolve into factors of cylic polynomials

There is no hard and fast rule for factorizing polynomials. Often a factor can be found by suitable rearranging the terms. Sometimes assumming the expression as polynomial of the variable, we determine one or more than one factors by the factor theorem and considering the cyclic and symmentc properties of the expression, we determine the remaining factors.

In this regard, it is mentionable that, of the variable a, b, c .

(a) If $(a-b)$ be the factor of a cyclic polynomial then $(b-c)$ and $c-a$ will be factor of that polynomial.

(b) $k(a+b+c)$ and $k(a^2+b^2+b^2)+m(ab+bc+ca)$ are the homogeneous and cyclic polynomial of degree 1 and 2 where *k* and *m* are constant..

(c) If the values of two polynomials are equal for all values of the variables, the coefficient of the corresponding two terms of the polynomial where k and m are the constants is equal.

Example 2. Resolve into factors $bc(b-c)+ca(c-a)+ab(a-b)$.

Solution : First Method :

$$bc(b-c)+ca(c-a)+ab(a-b)$$
$$=bc(b-c)+c^2a-ca^2+a^2b-ab^2$$
$$=bc(b-c)+a^2b-ca^2-ab^2+c^2a$$
$$=bc(b-c)+a^2(b-c)-a(b^2-c^2)$$
$$=(b-c)\{bc+a^2-a(b+c)\}$$
$$=(b-c)\{bc+a^2-ab-ac\}$$
$$=(b-c)\{bc-ab-ac+a^2\}$$
$$=(b-c)\{b(c-a)-a(c-a)\}$$
$$=(b-c)(c-a)(b-a)$$
$$=-(a-b)(b-c)(c-a).$$

Second Method : Considering the given expression as a polynomial $P(a)$ in a, we substitute b for a and $P(b)=bc(b-c)+cb(b-b)+b^2(a-b)=0$. So $(a-b)$ is a factors of the given expression. Similarly, $(b-c)$ and $(c-a)$ are factors of the given expression. The product of these three factors is a cyclic homogeneous polynomial of degree 3, as is the given expression. So, any remaining factor must be a constant. Suppolse $bc(b-c)+ca(c-a)+ab(a-b)=k(a-b)(b-c)(c-a)$ putting $a=0, b=1, c=2$ in this indetity we get $2(-1)=k(-1)(-1)(2)\Rightarrow -2=2k\Rightarrow k=-1$.

$\therefore\ bc(b-c)+ca(c-a)+ab(a-b)=-(a-b)(b-c)$.

Example 3. Factorize $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

Solution : Considering the given expression as a polynomial $P(a)$ in a, we substitute b for a and get, $P(b)=b^3(b-c)+b^3(c-b)+c^3(b-b)=0$. So, $(a-b)$ is a factor of the given expression. Similarly, $(b-c)$ and $(c-a)$ are factors of the given expression. The product of these three factors is a cyclic homogeneous polynomial of degree 3, while the given expression is a cyclic homogeneous polynomial of degree 4. So the remaining factor must be a cyclic homogeneous polynomial of degree 1. Upto multiplication by a non-zero constant, the only such polynomial is $a+b+c$. Therefore,

$\therefore\ a^3(b-c)+b^3(c-a)+c^3(a-b)=k(a-b)(b-c)(c-a)(a+b+c)$

Where k is a constant,
Putting $a = 0, b = 1, c = 2$ in this identity
we get, $2 + 8(-1) = k(-1)(-1)(2)(3) \Rightarrow$ -6 = 6k $\Rightarrow k = -1$
$\therefore\ a^3(b-c) + b^3(c-a) + c^3(a-b) = -(a-b)(b-c)(c-a)(a+b+c)$.

Example 4. Factorize $(b+c)(c+a)(a+b) + abc$.

Solution : Considering the expression as a polynomial $P(a)$ of a, we substitie $-b-c$ for a and get,
$P\{-(b + c)\} = (b + c)(c - b - c)(-b - c + b) + (-b-c)bc(b + c) - bc, (b + c) = 0.$
So, according to the factor theorem, $(a+b+c)$ is the factor of the given expression. The given expression having a cyclic homogeneous polynomial of degree 3 and one factor of degree 1 have been found. So, the remaining factor will be cyclic homogenous polynomial of degree two, i.e., will be of the form $k(a^2+b^2+c^2)+m(bc+ca+ab)$, there k and m are constants.

$\therefore\ (b+c)(c+a)(a+b)+abc = (a+b+c)\{k(a^2+b^2+c^2)m(bc+ca+ab)\}$.........$(i)$

For all values of $a, b, c,$ (i) is true.

Putting in (i) first $a = o, b = 0, c = 1$ and then $a = 1, b = 1, c = 0$, we get respectively $0 = k$ and $2 = 2(k \times 2 + m)$

$\therefore\ k = 0, m = 1$.

Now putting the values of k and m, we get, $(ab+c)(c+a)(a+b)+abc = (a+b+c)$ $(bc+ca+ab)$.

Remarks : The expressions described in example 3 and example 4 will be factorized by using the similar first method of solving example 2.

An Important Algebraic Formula

For all the values of a, b and c, two proofs of the formula are given below :

$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

First Proof (using simple algebraic manipulations) :

$$\begin{aligned}
&a^3+b^3+c^3-3abc \\
&= (a+b)^3 - 3ab(a+b) + c^3 - 3abc \\
&= (a+b)^3 + c^3 - 3ab(a+b+c) \\
&= (a+b+c)\{(a+b)^2 - (a+b)c + c^2\} - 3ab(a+b+c) \\
&= (a+b+c)(a^2+2ab+b^2-ac-bc+c^2) - 3ab(a+b+c) \\
&= (a+b+c)a^2+b^2+c^2-ab-bc-ca)
\end{aligned}$$

Second Proof (using properties of homogeneous cyclic expressions)

Considering the expression $a^3+b^3+c^3-3abc$ a polynomial $P(a)$ of the variable a and putting in it $a = -(b+c)$ we get, We get, $p\{-(b+c)\} = -(b+c)^3 + b^3 + c^3 - 3(b+c)bc$

$= -(b+c)^3 + (b+c)^3 = 0$

Therefore, $a-(-(b+c))$ = $a+b+c$ is a factor of the expression under consideration. As $a^3+b^3+c^3-3abc$ is a cyclic homogeneous cyclic polynomial of degree 3, the other factor will have the form $k(a^2+b^2+c^2)+m(ab+bc+ca)$, where k and m are constants. So, for all a,b,c

$a^3+b^3+c^3-3abc = (a+b+c)\{k(a^2+b^2+c^2) \ +m(ab+bc+ca)\}$

Putting successively $a=1, b=0, c=0$ and $a=1, b=1, c=0$ we get,

$1=k$ and $2=2(k\times 2+m) \Rightarrow k=1$ and $1=2+m \Rightarrow m=-1$

$\therefore$ $k=1$ and $m=-1$

$\therefore$ $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

Corollary 1 : $a^3+b^3+c^3-3abc = \frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$

Proof : Since $\frac{1}{2}\{2(a^2+b^2+c^2-ab-bc-ca)\}$

$= \frac{1}{2}\{(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca-a^2)\}$

$= \frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$.

$\therefore$ $a^3+b^3+c^3-3abc$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= \frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+c-a)^2\}$

Corollary 2 : If $a+b+c=0$, so $a^3+b^3+c^3=3abc$.

Corollary 3 : If $a^3+b^3+c^3=3abc$, so $a+b+c=0$ or $a=b=c$.

Example 5. Factorize $(a-b)^3+(b-c)^3+(c-a)^3$.

Solution : Suppose $A=a-b, B=b-c$ and $C=c-a$. Then

$A+B+C = a-b+b-c+c-a = 0$

Therefore, $A^3+B^3+C^3 = 3ABC$

That is, $(a-b)^3+(b-c)^3+(c-a)^3 = 3(a-b)(b-c)(c-a)$.

Activity :

1. Resolve into factors :

(i) $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$

(ii) $a^2(b-c)+b^2(c-a)+c^2(a-b)$

(iii) $a(b-c)^3+b(c-a)^3+c(a-b)^3$

(iv) $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$

(v) $a^4(b-c)+b^4(c-a)+c^4(a-b)$

(vi) $a^2(b-c)^3+b^2(c-a)^3+c^3(a-b)^3$

(vii) $x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)$

(viii) $a^3(b-c)+b^3(c-a)+c^3(a-b)$

2. If $\dfrac{x^2-yz}{a}=\dfrac{y^2-zx}{b}=\dfrac{z^2-xy}{c}\neq 0$,

show that $(a+b+c)(x+y+z)=ax+by+cz$.

3. If $(a+b+c)(ab+bc+ca)=abc$, show that $(a+b+c)^3=a^3+b^3+c^3$.

Rational Fractions

Fraction formed with a polynomial as denominator and a polynomial as numerator is called the rational fraction. For example, $\dfrac{x}{(x-a)(x-b)}$ and $\dfrac{a^2+a+1}{(a-b)(a-c)}$ are the rational fractions.

Example 1. Simplify $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$

Solution : Given expression $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$

$$=\frac{a}{-(a-b)(a-c)}+\frac{b}{-(b-c)(b-a)}+\frac{c}{-(c-a)(c-b)}$$

$$=\frac{a(b-c)+b(c-a)+c(a-b)}{-(a-b)(b-c)(c-a)}$$

$$=\frac{ab-ca+bc-ab+ca-bc}{-(a-b)(b-c)(c-a)}$$

$$=\frac{0}{(a-b)(b-c)(c-a}$$

$$=0$$

Example 2. Simplify : $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}+\dfrac{b^2-(a-c)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$

Sulution : 1^{st} fraction = $\dfrac{(a+b-c)(a-b+c)}{(a+b+c)(a+b-c)}=\dfrac{a+b-c}{a+b+c}$

2^{nd} fraction = $\dfrac{(b+c-a)(b-a+c)}{(a+b+c)(a+b-c)}=\dfrac{b-a+c}{a+b+c}$

3^{rd} fraction = $\dfrac{(c+a-b)(c-a+b)}{(b+c+a)(b+c-a)}=\dfrac{c+a-b}{a+b+c}$

$\therefore$ Given expression = $\dfrac{a+b-c}{a+b+c}+\dfrac{b-a+c}{a+b+c}+\dfrac{c+a-b}{a+b+c}$

$$=\frac{a+b-c+b-a+c+c+a-b}{a+b+c}$$

$$=\frac{a+b+c}{a+b+c}=1$$

Example 3. Simplify $\dfrac{(ax+1)^2}{(x-y)(z-x)}+\dfrac{(ay+1)^2}{(x-y)(y-z)}+\dfrac{(az+1)^2}{(y-z)(z-x)}$

Solution : Given expression

$$=\frac{(ax+1)^2(y-z)+ay+1)^2(z-x)+(az+1)^2(x-y)}{(x-y)(y-z)(z-x)}\text{.......(1)}$$

$$=(a^2x^2+2ax+1)(y-z)+(a^2y^2+2ax+1)(z-x)+a^2z^2+2ax+1)(x-y)$$

$$=a^2\{x^2(y-z)+y^2(z-x)+z^2(x-y)\}+2\{x(x(y-z)+y(z-x)+z(x-y)\}$$

$$+\{y-z)+(z-x)+(x-y)\}$$

but, $x^2(y-z)+y^2(z-x)+z^2(x-y)=-(x-y)(y-z)(z-x)$

and, $x(y-z)+y(z-x)+z(x-y)=0$ Ges $(y-z)+(z-x)+(x-y)=0$

$\therefore$ numerator of (1) = $-a^2(x-y)(y-z)(z-x)$

Therefore the required expression = $\dfrac{-a^2(x-y)(y-z)(z-x)}{(x-y)(y-z)(z-x)}=a^2$

Exampe 4. Simplify $\dfrac{1}{x+a}+\dfrac{2x}{x^2a^2}+\dfrac{4x^3}{x^4a^4}+\dfrac{8x^7}{a^8-x^8}$

Solution : The sum of the given 3rd and 4th terms

$$= \frac{4x^3}{x^4+a^4} + \frac{8x^7}{a^8-x^8} = \frac{4x^3}{x^4+a^4}\left(1+\frac{2x^4}{a^4-x^4}\right)$$

$$= \frac{4x^3}{x^4+a^4} \times \frac{a^4-x^4+2x^4}{a^4-x^4} = \frac{4x^3}{x^4+a^4} \times \frac{a^4+x^4}{a^4-x^4}$$

$\therefore$ the sum of a 2nd, 3rd and 4th terms

$$= \frac{2x}{x^2+a^2} + \frac{4x^3}{a^4-x^4} = \frac{2x}{x^2+a^2}\left[1+\frac{2x^2}{a^2-x^2}\right]$$

$$\frac{2x}{x^2+a^2} \times \frac{a^2-x^2+2x^2}{a^2-x^2} = \frac{2x}{x^2+a^2} \times \frac{a^2+x^2}{a^2-x^2} = \frac{2x}{a^2-x^2}$$

$\therefore$ the given expression $= \dfrac{1}{x+a} + \dfrac{2x}{a^2-x^2} = \dfrac{a-x+2x}{a^2-x^2} = \dfrac{a+x}{a^2-x^2} = \dfrac{1}{a-x}$.

Activity : Simplify

1. $\dfrac{b+c}{(a-b)(a-c)} + \dfrac{c+a}{(b-c)(b-a)} + \dfrac{a+b}{(c-a)(c-b)}$

2. $\dfrac{a^3-1}{(a-b)(a-c)} + \dfrac{b^3-1}{(b-c)(b-a)} + \dfrac{c^3-1}{(c-a)(c-b)}$

3. $\dfrac{bc(a+d)}{(a-b)(a-c)} + \dfrac{ca(b+d)}{(b-c)(b-a)} + \dfrac{ab(c+d)}{(c-a)(c-b)}$

4. $\dfrac{a^3+a^2+1}{(a-b)(a-c)} + \dfrac{b^3+b^2+1}{(b-c)(b-a)} + \dfrac{c^3+c^2+1}{(c-a)(c-b)}$

5. $\dfrac{a^2+bc}{(a-b)(a-c)} + \dfrac{b^2+ca}{(b-c)(b-a)} + \dfrac{c^2+ab}{(c-a)(c-b)}$

2.4 Partial Fractions

If a given fraction is expressed as a sum of two or more fractions, then each of the latter fractions is said to be a partial fraction of the given fraction.

For example, the fraction $\dfrac{3x-8}{x^2-5x+8}$ can be written down,

$$\frac{3x-8}{x^2-5x-8} = \frac{2(x-3)+(x-2)}{(x-3)(x-2)} = \frac{2}{x-2} + \frac{1}{x=3}$$

There the given expression is expressed as the sum of the two expressions i.e., the fraction has been divided into two partial fraction.

If both $N(x)$ and $D(x)$ are the polynomials of the variable x and if the degree of the numerator $N(x)$ is less that of the denominator $D(x)$, that fraction is called a proper fraction. If the degree of the numerator $N(x)$ is greater than or equal to that of the denominator $D(x)$, the fraction is called an improper fraction. For example, $\frac{x^2+1}{(x+1)(x+2)(x-3)}$ is a proper fraction, while $\frac{2x^4}{x+1}$ and $\frac{x^3+3x^2+2}{x+2}$ are both improper fractions.

We mention that numerator divided by the denominator of the improper fraction in general rule or the terms of the numerator rearranged in a convenient way, the fraction as a polynomial (quotient) and as the sum of the improper fraction can be expressed.

For exameple, $\frac{x^3+3x^2+2}{x+2} = (x+x-2)+\frac{6}{x+2}$.

How to convert the proper rational fraction into the partial fraction in different ways is shown below :

Case 1 : The denominator is, or can be expressed as a product of distrinct linear fractors but no factor is repeated.

Example 1. Express : $\frac{5x-7}{(x-1)(x-2)}$ partial fractions.

Solution : Suppose $\frac{5x-7}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2}$ …….. (i)

Multiplying both sides of (i) with $(x-1)(x-2)$ we get

$$5x-7 = A(x-2)+B(x-1) \text{ ……….. (ii)}$$

which is true for all values of x. Putting $x=1$. In both sides of (ii) we get,

$5-7 = A(1-2)+B(1-1)$

or, $-2=-A \;\therefore\; A=2$

Again, putting $x=2$ in both sides of (ii) we get, $10-7 = A(2-2)+B$

or, $3=B \;\therefore\; 3 = B$

Now putting the values of A and B in (i) we get,

$\frac{5x-7}{(x-1)(x-2)} = \frac{2}{x-1}+\frac{3}{x-2}$; thus the given fraction is converted into partial fractions.

Remark : That the given fraction is converted into the partial fraction properly, can be examined.

R.H.S. $\frac{2}{x-1}+\frac{3}{x-2} = \frac{2(x-2)+3(x-1)}{(x-1)(x-2)} = \frac{5x-7}{(x-1)(x-2)}$ = L.H.S.

Example 2. Express $\dfrac{x+5}{(x-1)(x-2)(x-3)}$ as partial fractions.

Solution : Suppose, $\dfrac{x+5}{(x-1)(x-2)(x-3)} \equiv \dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$.......$(i)$

Multiply both sides (i) by $(x-1)(x-2)(x-3)$ we get,

$$x+5 \equiv A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)........(ii)$$

The both sides of (ii) are true for all values of x.

Puttiing $x=1$ in both sides of (ii) we get, $1+5=A(-1)(-2) \Rightarrow 6=2A \Rightarrow A=3$ ..... (ii)

Again, Putting $x=2$ in both sides of (ii) we get,

$2+5=B(1)(-1) \Rightarrow 7=-B \quad \therefore \ B=-7$

And Putting $x=3$ in both sides of (ii) we get, $3+5=C(2)(1)$ or, $8=2c$ or $c=4$

Now, Putting the values of A, B and C in (i) we get,

$\dfrac{x+5x}{(x-1)(x-2)(x-3)}=\dfrac{3}{x-1}-\dfrac{7}{x-2}+\dfrac{4}{4-3}$ This is the expression of the given fraction into the partial fraction.

Case 2 : The denominator is, or can be expressed as a product of distinct linear factors, but the degree of the numerator is greater than or equal to that of the denominator.

Example 3. Express $\dfrac{(x-1)(x+5)}{(x-2)(x-4)}$ as partial fractions.

Solution : Suppose, $\dfrac{(x-1)(x+5)}{(x-2)(x-4)} \equiv 1+\dfrac{A}{x-2}+\dfrac{2}{2-4}$ .... (i)

Multiplying both sides of (i) with $(x-2)(x-4)$ we get,

$$(x-1)(x-5) \equiv (x-2)(x-4)+A(x-4)+B(x-2) \text{ .... (ii)}$$

Putting successively $x=2$ and $x=4$ in both sides of (ii) we get,

(2 – 1)(2 – 5) = and (2 – 4) or, $A=\dfrac{3}{2}$

and $(4-1)(4-5)=B(4-2)$ or, $B=\dfrac{-3}{2}$

Now, putting the values of A and B in (i) we get,

$\dfrac{(x-1)(x+5)}{(x-2)(x-4)}=1+\dfrac{3}{2(x-3)}-\dfrac{3}{2(x-4)}$ which is the desired partial fractions.

Example 4. Express $\dfrac{x^3}{(x-1)(x-2)(x-3)}$ as partial fractions.

Solution : Suppose $\dfrac{x^3}{(x-1)(x-2)(x-3)} = 1 + \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$ .... (i)

Multiplying both sides of (i) with $(x-1)(x-2)(x-3)$ we get,

$x^3 = (x-1)(x-2)(x-3) + A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$ ..... (ii)

Putting successively $x=1, x=2$ and $x=3$ in the both sides of (ii) we get,

$1 = A(-1)(-2)$ or, $A = \dfrac{1}{2}$

$8 = B(1)(-1)$ or, $B = -8$

And $27 = C(2)(1)$ or, $C = \dfrac{27}{2}$

Now, putting the valus of *A, B* and *C* in (i), we get,

$\dfrac{x^3}{(x-1)(x-2)(x-3)} = 1 + \dfrac{1}{2(x-1)} - \dfrac{8}{x-2} + \dfrac{27}{2(x-3)}$ which is the desired partial fractions.

Case 3 : The denominator is, or can be expressed as a product of linear factors, some of which are repeated.

Example 5. Express $\dfrac{x}{(x-1)^2(x-2)}$ as partial fractions.

Solution : Suppose $\dfrac{x}{(x-1)^2(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2}$.........$(i)$

Multiplying both sides of (i) with $2(x-1)(x-2)$ we get

$$x = A(x-1)(x-2) + B(x-2) + C(x-1)^2 \text{ ......(ii)}$$

Putting successively $x=1$ and $x=2$, in the both sides of (ii) we get,

$1 = B(1-2)$ or, $B = -1$

And $2 = C(2-1)^2$ or, $2 = C \Rightarrow C = 2.$

Equating the coefficients of x^2 on both sides of (i)

$0 = A + C = A = -C = -2$.

Now putting the values of *A, B* and *C* in (i), we get,

$\dfrac{x}{(x-1)^2(x-2)} = \dfrac{-2}{x-1} + \dfrac{-1}{(x-1)^2} + \dfrac{2}{x-2}$ which is the desired partial fractions.

Case 4 : The denominator is or can be expressed as a product linear and quadratic factor, none of which is repeated.

Example 6. Express $\dfrac{x}{(x-1)(x^2+4)}$ as partial fractions.

Solution : Suppose $\dfrac{x}{(x-1)(x^2+4)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4}$ .... (i)

Multipying both sides of (i) with $(x-1)(x^2+4)$ we get,

$x = A(x^2+4)+(Bx+C)(x-1)$ .... (ii)

Putting $x=1$ in (ii) we get, $1 = A(5) \Rightarrow A = \dfrac{1}{5}$

Equating coefficients of x^2 and of x we get, $A+B=0$ .... (iii)
and $C-B=1$ .... (iv)

Putting $A=\dfrac{1}{5}$ (iii) we get, $B=-\dfrac{1}{5}$

Putting $B=-\dfrac{1}{5}$ (iv) we get, $C=\dfrac{4}{5}$

Now putting the values of *A, B* and *C* in (i) we get,

$$\frac{x}{(x-1)(x^2 4)} = \frac{\frac{1}{5}}{x-1} + \frac{\frac{-x}{5}+\frac{4}{5}}{x^2-4} = \frac{1}{5(x-1)} - \frac{x-4}{4(x^2-4)}$$

which is the desired partial fractions.

Case 5 : The denominator is, or can be expressed as a product of linear and quadratic factors, some of which are repeated.

Example 7. Express $\dfrac{1}{x^2(x^2+1)^2}$ as partial fractions.

Solution : Ssuppose $\dfrac{1}{x^2(x^2+1)^2} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$ .... (i)

Multiplying both sides of (i) with $x^2(x^2+1)^2$ we get,

$$1 = Ax(x^2+1)^2 + B(x^2+1)^2 + (Cx+D)x^2(x^2+1) + Ex+F)x^2 \text{ .... (ii)}$$

Equating coefficients of x^5, x^4, x^3, x^2, x and the constant terms of (ii) we get, respectively,

$A+C=0,\quad B+D=0,\quad 2A+C+E=0,\quad 2B+D+F=0$

$\therefore\ A=0, B=1.$

$\Rightarrow C=0,\quad D=-1,\quad E=-2A-C=0,\quad F=-2B-D=-2+1=-1$

Putting $A=0$ in $A+C=0$, we get, $C=0$

Putting $B=1$ in $B+D=0$, we get, $D=-1$

Again, Putting $A=0, C=0$ in $2A+C+E=0$, we get, $E=0$

$\therefore$ Putting in $A=0, B=1, C=0, D=-1, E=0, F=-1$ we get,

$$\frac{x}{x^2(x^2+1)^2}=\frac{1}{x^2}-\frac{0-1}{x^2+1}+\frac{0-1}{(x^2+1)^2}=\frac{1}{x^2}-\frac{1}{x^2+1}-\frac{1}{(x^2+1)^2}.$$

is the desired expresstion of the given fraction as a sum of partial fractions.

Activity : Express as a sum of partial fraction :

1. $\dfrac{x^2+x-1}{x^3+x^2-6x}$ 2. $\dfrac{x^2}{x^4+x^2-2}$ 3. $\dfrac{x^3}{x^4+3x^2+2}$

4. $\dfrac{x^2}{(x-1)^3(x-2)}$ 5. $\dfrac{1}{1-x^3}$ 6. $\dfrac{2xdx}{(x+1)(x^2+1)^2}$

Exercise 2

1. Which one of the following expression is symmetric ?
   (i) $a+b+c$ (ii) $xy+yz+zx$
   (iii) $x^2-y^2+z^2$ (iv) $2a^2-5bc-c^2$
2. (i) If $a+b+c=0$ then $a^3+b^3+c^3=3abc$
   (ii) The expression $P(x,y,z)=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$ is cyclic
   (iii) The simplified form of $\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{x^4-1}$ is $\frac{1}{x-1}$.

Which one of the following statements is correct ?
   (a) i and ii (b) ii and iii (c) i and iii (d) i, ii and iii

   Given that $x+7$ is a factor of the polynomial x^3+px^2-x-7. answer the following questions 3 and 4 below :

3. What is the value of P ?
   (a) -7 (b) 7 (c) $\frac{54}{7}$ (d) 477
4. What is product of the other factors of the polynomial ?
   (a) $(x-1)(x-1)$ (b) $(x+1)(x-2)$ (c) $(x-1)(x+3)$
   (d) $(x+1)(x-1)$
5. If $x-2$ is a factor of the polynomial $x^4-5x^3+7x^2-a$, show that $a=4$.
6. Let $P(x)=x^n-a^n$, where n is a positive integer and a is a constant.
   (i) Show that $(x-a)$ is a factor of the polynomial. and find $Q(x)$ such that $P(x)=(x-a)Q(x)$ holds.
   (ii) If n is an even number, show that $(x+a)$ is a factor of the polynomial and find $Q(x)$ such that $P(x)=(x+a)Q(x)$ holds.

7. Let $P(x)=x^n+a^n$, where n is a positive integer and a is a constant. If n is an odd number, show that $(x+a)$ is a factor of the polynomial and find $Q(x)$ such that $P(x)=(x+a)Q(x)$ holds.

8. Let $P(x)=ax^5+bx^4+cx^3+x^2+bx+a$ where a,b,c are constants and $a\neq 0$. If $(x-r)$ is a factor of $P(x)$, show that $(rx-1)$ too will be a factor of $P(x)$.

9. Resolve into factors :

   (i) $x^4+7x^3+17x^2+17x+6$

   (ii) $4a^4+12a^3+7a^2-3a-2$

   (iii) x^3+2x^2+2x+1

   (iv) $x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)+3xyz$

   (v) $(x+1)^2(y-z)+(y+1)^2(z-x)+(z+1)^2(x-y)$

   (vi) $b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2)+a^2b^2(a^2-b^2)$.

10. If $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{3}{abc}$, show that, $bc+ca+ab=0$ or $a=b=c$.

11. If $x=b+c-a, y=c+a-b$ and $z=a+b-c$, show that

    $x^3+y^3+z^3=4(a^3+b^3+c^3-3abc)$

12. Simplify :

    (i) $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$

    (ii) $\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}$

    (iii) $\frac{(a+b)^2-ab}{(b-c)(a-c)}+\frac{(b+c)^2-bc}{(c-a)(b-a)}+\frac{(c+a)^2-ca}{(a-b)(c-b)}$

    (iv) $\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\frac{8}{1+x^8}+\frac{16}{x^{16}-1}$

13. Express as a sum of partial fractions :

    (a) $\frac{5x+4}{x(x+2)}$ (b) $\frac{x+2}{x^2-7x+12}$ (c) $\frac{x^2-9x-6}{x(x-2)(x+3)}$

    (d) $\frac{x^2-4x-7}{(x+1)(x^2+4)}$ (e) $\frac{x^2}{(2x+1)(x+3)^2}$

14. The polynomial of the variable x is $P(x)=7x^2-3x+4x^4-a+12x^3$;

    (a) Write down the standard form of $P(x)$ and give an example of a reverse polynomial of third degree.

(b) Find the value of a if $(x+2)$ is a factor of the polynomial $P(x)$.

(c) If $Q(x)=6x^3-x^2+9x+2$, and $Q\left(\frac{1}{2}\right)=0$, find the two common factors of $P(x)$ and $Q(x)$.

15. The polynomial of x, y, z is $F(x,y,z)=x^2+y^2+y^2-2xyz$;

(a) Show that $F(x,y,z)$ is a cyclic expression.

(b) Factorize $F(x,y,z)$ and show that if $F(x,y,z)=0$, $x+y+z\neq 0$, so $(x^2+y^2+z^2)=(xy+yz+zx)$.

(c) If $x=b+c-a, y=c+a-b$ and $z=a+b-c$, show that $F(a,b,c):F(x,y,z)=1:4$.

16. The four expressions of the variable x are $(x+x),(x^2-9),(x^3+27)$ and (x^4-81) :

(a) Form a proper rational fraction and an improper rational fractions from the above expressions.

(b) Express $\frac{x^2+27}{x^2-9}$ as a sum of partial fractions.

(c) Simplify the sum of the reciprocals (multiplicative inveses) of the first, second and fourth of the above expressions.

17. The expression $(x+1)^3y+(y+1)^2$:

(a) Express it as a polynomial of the variable x in its standard form, and find its degree, leading coefficient and constant term (term independent of x).

(b) Express it as a polynomial of the variable y in its standard form, and find its degree, leading coefficient and constant term.

(c) Find its degree as a polynomial of the variables x and y.

Chapter Three
Geometry

In Geometry of secondary and lower secondary and lower secondary level, the theorem of Pythagoras and its converse have been discussed in detail. In learning mathematics, the subjects related to Pythagoras play an important role. So, in light of the theorem of Pythagoras, more discussion is necessary in Secondary Higher Mathematics. For this discussion, it needs to have a clear conception of orthogonal projecton. With this objective in view, in the first part of this stage a brief discussion of the theorem of Pythagoras, at the second stage the conception of orthogonal projection and the corollary of the theorem will be discussed. In the concluding part of the discussion, on the basis of Pythagoras and its extended conception, some problems will be included for logical discussion and proof.
At the end of this chapter, the students will be able to –

- Explain the conception of orthogonal projection
- Prove and apply the theorems on the basis of the theorem of Pythagoras
- Prove and apply the theorems of the circumcentre, centroid and orthocenter
- Prove and apply the theorems of Brahmagupta
- Prove and apply the theorems of Pytolemy.

3. (a) Review of the Theorem of Pythagoras

About 600 years before the birth of Christ, the celebrated Greek scholar Pythagoras described an extremely important theorem about right angled triangles. This theorem is known as the theorem of Pythagoras as it is named after him. But it is 1000 years before that; it is known that the Egyptian surveyors had the knowledge about the theorem. The theorem of Pythagoras can be proved in many ways. There are its two proofs in lower secondary level. Here only its description and some discussion will be placed.

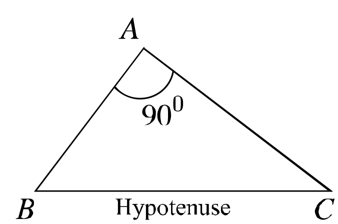


Fig : 3.1 Right Angled Triangle

Theorem 3.1

Theorem of Pythagoras

In a right angled triangle the area of the square drawn on the hypotenuse is equal to the sum of the areas of the two squares drawn on the other two sides.

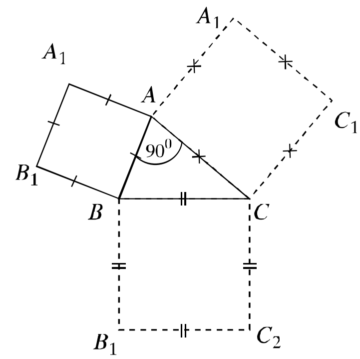


Fig : 3.2

In the figure 3.2, ABC is a right angled triangle. $\angle BAC$ is a right angle and BC is hypotenuse. If any square is drawn on the hypotenuse BC, its area is equal to the sum of the areas of the squares drawn on the sides adjacent to the right angle, AB and AC.

That is, $BC^2 = AB^2 + AC^2$

Here $BC^2 =$ the area of the square BB_1C_2C

$AB^2 =$ " " " " " AA_1B_1B

$AC^2 =$ " " " " " AA_1C_1C

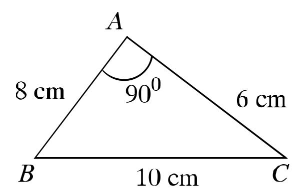


Fig : 3.3

For example (in figure 3.3), if the lengths of the two sides adjacent to the right angle of a right angled triangle are 8 cm. and 6 cm. respectively, it is said easily as per the theorem of Pythagoras that the length of the hypotenuse will be 10 cm.

Similarly, it is possible to know the length of the third side by the lengths of any two sides. The following theorem is known as the converse theorem of Pythagoras.

Theorem 3.2

It the area of the square on one side in a triangle is equal to the sum of areas the squares drawn on the other two sides, the angle included by the latter two sides is a right angle. Look at (the figure 3.4)

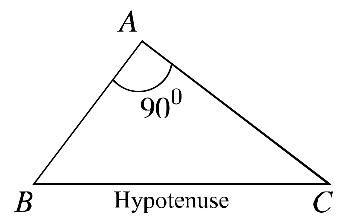


Fig : 3.4

The side BC of ΔABC is a hypotenuse and the other two sides are AB and AC respectively. The area of the square drawn on the side BC is equal to the sum of the sides AB and AC respectively.

That is, $BC^2 = AB^2 + AC^2$

So, $\angle BAC$ is a right angle

For example, we can say that if the lengths AB, BC and CA of ΔABC are 6 cm., 10 cm. and 8 cm. respectively, $\angle BAC$ must be right angle.

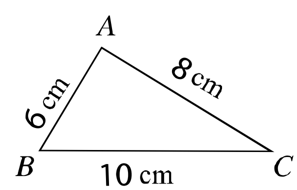


Fig : 3.5

Since, $AB^2 = 6^2$ sq. cm. $= 36$ sq. cm.

$BC^2 = 10^2$ sq. cm. = 100 sq. cm.

$AC^2 = 8^2$ sq. cm. = 64 sq. cm.

$\therefore\ BC^2 = 100 = 36 + 64 = AB^2 + AC^2$.

$\therefore\ \angle BAC = 90^\circ$ = right angle

3 (b) Orthogonal Projection

We call the arthogonal projection of any point on any definite straight line when it signifies the foot of the perpendicular drawn from that point on the definite straight Line.

Suppose, *XY* is a definite straight line and *P* is any point (figure 3.6). We dray the perpendicular *PP′* from *P* on the line *XY* and *P* on the line *XY* and *P′* is the foot of the perpendicular *PP′* So, the point *P′* is the orthogonal of *P* on the line *XY*. That is, the orthogonal projection of any point on any definite straight line is a point. From this conception we can say that the perpendicular on any straight line is a point on any straight line of the orthogonal projection. In that case, the length of that orthogonal projection will be zero.

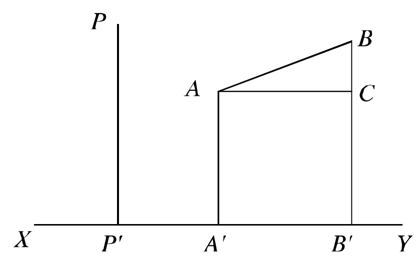


Fig : 3.6

Orthogonal Projection of a Line Segment

Let, the and points of the line segment *AB* are *A* and *B* (figure 3.6). Now the perpendiculars *AA′* and BB′ respectively are drawn on the line *XY* from the points *A* and *B*. *A′* is the foot of the perpendicular *AA′* and *B′* is the foot point of the perpendicular *BB′*. So, the line segment *A′B′* is the orthogonal projection of the line segment *AB* on *XY*.

Therefore, it is seen that the orthogonal projection is determined by drawing perpendicular. So, it is said that the line segment *A′B′* is the orthogonal projection of the line segment *AB* on *XY*.

Noted:

1. The foot of the perpendicular drawn from any point on any line is the orthogonal projection of that point.
2. The perpendicular of the orthogonal projection on any line is a point. So, the length of the orthogonal projection is zero.
3. Any definite line parallel to the line segment of the orthogonal projection will be equal to that line segment.

In the figure 3.6, if the line segment *AB* is parallel to *XY*, then *AB* = *A′B′*

Some Important Theorems

We shall present the logical proof of some important theorems on the basis of the theorem of Pythagoras and by the conception of the orthogonal projection.

Theorem 3.3

The area of the square drawn on the opposite side of the obtuse angle of an obtuse angled triangle is equal to the sum of the two squares drawn on the other two sides multiplied by twice the area of the rectangle included by anyone of the two other sides and the orthogonal projection of the other side on that side.

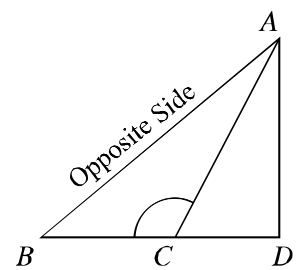


Fig : 3.7

Particular Enunciation: Suppose, in the triangle *ABC*, $\angle BCA$ is an obtuse angle, *AB* is the opposite side of the obtuse angle and the sides adjacent to obtuse angle are *BC* and *AC* respectively.

The side *CD* is the orthogonal projection of the side *AC* on the extended side *BC* (figure 3.7). It is to be proved that,

$AB^2 = AC^2 + BC^2 + 2BC \cdot CD.$

Proof: As the side *CD* is the orthogonal projection of the side *AC* on the extended side *BC*, ΔABD is a right angled triangle and $\angle ADB$ is right angle.

So, according to the theorem of Pythagoras

$$AB^2 = AD^2 + BD^2$$
$$= AD^2 + (BC + CD)^2 \qquad [\because BD = BC + CD]$$
$$= AD^2 + BC^2 + CD^2 + 2BC \cdot CD.$$

$\therefore AB^2 = AD^2 + CD^2 + BC^2 + 2BC \cdot CD$..........(1)

Again ΔACD is a right angled triangle and $\angle ADC$ is right angle

$\therefore AC^2 = AD^2 + CD^2$............(2)

From the equation (2), putting the value of AC^2 in equation (1), we get,

$AB^2 = AC^2 + BC^2 + 2BC \cdot CD$ [proved]

Theorem 3.4:

In any triangle, the area of the square drawn on the opposite side of an acute angle is equal to the squares drawn on the other two sides diminished by twice the area of the rectangle included by any

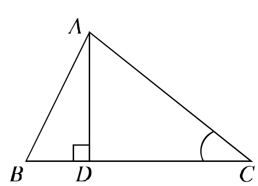


Fig : 3.8 (a)

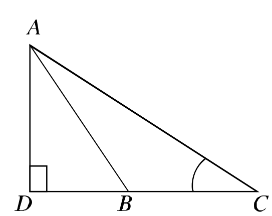


Fig : 3.8 (b)

one of the other sides and the orthogonal projection of the other side on that side.

Particular Enunciation : Suppose, in the triangle *ABC*, $\angle C$. is an acute angle and the opposite side of the acute angle is *AB*. The other two sides are *AC* and *BC* respectively. Suppose, *AD* is a perpendicular on the side *BC* (figure : 3.8-b) and on the extended side of *BC* (figure 3.8-a). So, *CD* is the orthogonal projection of the side *AC* on the side *BC* in the case of both triangle.

Proof : $\angle ADB$ is right angle of ΔABC

$\therefore AB^2 = AD^2 + BD^2$ [the theorem of Pythagoras] .... (1)

In the first figure $BD = BC - DC$

In the second figure $BD = DC - BC$

$\therefore$ So, in both cases $BD^2 = (BC - DC)^2 = (DC - BC)^2$

$$= BC^2 + DC^2 - 2BC \cdot DC$$

$$= BC^2 + CD^2 - 2BC \cdot CD \quad [CD = DC]$$

$\therefore BD^2 = BC^2 + CD^2 - 2BC \cdot CD$..........(2)

Now from equation(i) and (ii), we get

$AB^2 = AD^2 + BC^2 + CD^2 - 2BC \cdot CD$

or, $AB^2 = AD^2 + CD^2 + BC^2 - 2BC \cdot CD$ ....(3)

Again ΔADC is a right angled triangle and $\angle D$ is right angle

$\therefore AC^2 = AD^2 + CD^2$ [the theorem of Pythagoras] .... (4)

From the equation (3) and (4), we get

$AB^2 = AC^2 + BC^2 - 2BC \cdot CD.$ [proved]

N.B. : By drawing a perpendicular on AB from the point C, the theorem is to be proved is the same way. The theorem 3.3 and the theorem 3.4 are established on the basis of the theorem 3.1. So the theorem 3.3 and the theorem 3.4 can be called the corollary of the theorem 3.1 i.e. the theorem of Pythagoras. Accepted corollary on the basis of the above discussion :

Noted :

The observation and the decisions on the basis of the theorem of Pythagoras and the orthogonal projection from the mentioned theorems.

1. In the case of the right angled triangle, the sides adjacent to the right angle are mutually perpendiculars, so each of their orthogonal protection is zero. Therefore, *BC.CD=0*.
2. In case *C* is righ angle, the sides *AC* and *BC* are mutually perpendicular. So, *CD=0* and *CE=0*; so Theorem 3.3 and 3.4 indeed are extensions of the Theorem of Pythagoras.

In the case of ΔABC

1. If $\angle C$ is an obtuse angle, $AB^2 > AC^2 + BC^2$ [Theorem 3·3]
2. $\angle C$ is an right angle, $AB^2 = AC^2 + BC^2$ [Theorem 3·1]
3. $\angle C$ is an acute angle, $AB^2 < AC^2 + BC^2$ [Theorem 3·4]

The following theorem is the of the theorem i.e. it is established on the basis of the theorem 3.3 and the theorem 3.4. As the theorem is stated by Apollonius, it is called the theorem of Apollonius Theorem.

Theorem 3.5 (Theorem of Apollonius) : The sum of the areas of the squares drawn on any two sides of a triangle is equal to twice the sum of area of the squares drawn on the median of the third side and on either half of that side.

Particular Enunciation : The median AD of ABC bisect the side BC. It is to be proved that: $AB^2 + AC^2 = 2(AD^2 + BD^2)$

Construction : We draw a perpendicular *AE* on the side *BC* (figure 3.9 (a) and on the extended side of *BC* (figure 3.9 (a)

Proof :

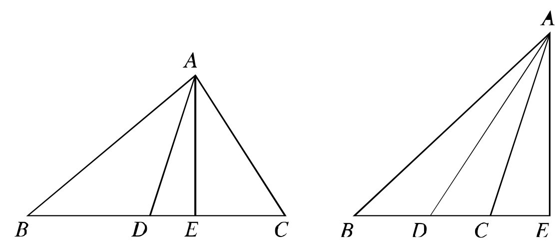


Fig : 3.9 (a) Fig : 3.9 (b)

ΔABD is an obtuse angle of $\angle ADB$ and *DE* is the orthogonal projection of the line *AD* on the *BD* extended (in both figures). As per the extension of the theorem of Pythagoras we get, $AB^2 = AD^2 + BD^2 + 2\,BD \cdot DE$..........(1)

Again, ΔACD is an acute angle of $\angle ADC$ and *DE* is the orthogonal projection of the line *AD* on the line *DC* (in the figure 3.9 (a) and on the line *DC* extended (in the figures 3.9 (b).

$\therefore$ As per the extension of the theorem of Pythagoras (the theorem 3.4) in the case of the acute angle. We get, $AC^2 = AD^2 + CD^2 - 2CD \cdot DE$...........(2)

Now adding the equations (i) and (ii), we get,

$$AB^2 + AC^2 = 2AD^2 + BD^2 + CD^2 + 2BD \cdot DE - 2CD \cdot DE$$
$$= 2AD^2 + BD^2 + BD^2 + 2BD \cdot DE - 2BD \cdot DE \; ; \quad [\; BD = CD]$$
$$= 2AD^2 + BD^2$$
$$= 2(AD^2 + BD^2). \qquad \text{[proved]}$$

Corollary : Determination of the relation between the side of the triangle and the median by the theorem of Apollonius.

Let *a,b,c* be the lengths of the sides *BC,CA,AB*, respectively of the triangle *ABC*. *d,e,f* are the lengths of the three medians *AD,BE,CF* respectively, Then from Apollonius's Theorem we get,

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

Or, $c^2 + b^2 = 2\left(d^2 + \left(\frac{1}{2}a\right)^2\right) \quad \left[\because BD = \frac{1}{2}a\right]$

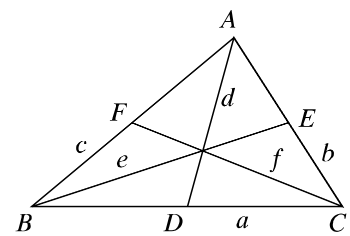
Or, $b^2 + c^2 = 2d^2 + 2\cdot\frac{1}{4}a^2$

Or, $b^2 + c^2 = 2d^2 + \frac{a^2}{2}$

Or, $d^2 = \frac{2(b^2 + c^2) - a^2}{4}$

$$e^2 = \frac{2(c^2 + a^2(-b^2}{4}$$

And $f^2 = \frac{2(a^2 + b^2) - c^2}{4}$



$\therefore$ If the lengths of the sides of any triangle are known, the length of the medians are to be known.

$$d^2 + c^2 + f^2 = \frac{2(b^2 + c^2 - a^2}{4} + \frac{2(c^2 + a^2) - b^2}{4} + \frac{2(a^2 + b^2) - c^2}{4}$$

$$= \frac{3}{4}(a^2 + b^2 + c^2)$$

$\therefore 3(a^2 + b^2 + c^2) = 4(d^2 + c^2 + f^2).$

So, we can say, in a right angled triangle, thrice the area of the square drawn on the hypotenuse is equal to twice the sum of the areas of the squares drawn on the three medians.

If the triangle is right angled i.e. C=1 right angle and *AB* is a hypotenuse,

$$c^2 = a^2 + b^2$$

$\therefore \quad a^2 + b^2 + c^2 = 2c^2$

or, $\frac{4}{3}(d^2 + c^2 + f^2) = 2c^2$

or, $2(d^2 + c^2 + f^2) = 3c^2.$

So, we can say, in a right angled triangle, thrice the area of the square drawn on the hypotenuse is equal to twice the sum of the areas of the squares drawn on the three medians.

Exercise 3.1

1. In ΔABC, the angle $\angle B = 60^\circ$; prove that, $AC^2 = AB^2 + BC^2 - AB \cdot BC$

2. In ΔABC, the angle $\angle B = 120^\circ$; prove that, $AC^2 = AB^2 + BC^2 - AB \cdot BC$

3. In ΔABC, the angle $\angle B = 90^\circ$; mid point of BC is D then prove that,
$AB^2 = AD^2 + 3BD^2$

4. In ΔABC, AD is the perpendicular on the side BC and BE the perpendicular on the side AC, show that, $BC \cdot CD = AC \cdot CE$

5. In ΔABC, the side BC is trisected at the point P and Q, proved that,
$AB^2 + AC^2 = AP^2 + AQ^2 + 4PQ^2$.
[Hints : $BP = BQ = AC$; AP is a median of ΔABQ.
$AB^2 + AQ^2 = 2 \cdot (BP^2 + AP^2) = 2PQ^2 + 2AQ^2$
AQ is a median of ΔAPC
$AP^2 + AC^2 = 2PQ^2 + 2AQ^2$]

6. In ΔABC, $AB = AC$, P is any point on BC. Prove that,
$AB^2 - AP^2 = BP \cdot PC$.
[Hints : Draw a perpendicular AD on BC, then
$AB^2 = BD^2 + AD^2$ and $AP^2 = PD^2 + AD^2$]

7. If the three medians of ΔABC meet at G, prove that
$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$

[Hints : See the corollaries taken in the light of the theorem of Apollonius i.e. therelation between the sides and the medians of the triangle will be seen.]

3.(c) Theorems about Circles and Triangles.

In this section some important theorems about circles and triangles will be presented logically. The students need to know about the similarity of the triangles before proving the theorems. For their proofs it is necessary to have knowledge about similarity of triangles. Similarly of triangles is discussed thoroughly in Secondary Geometry. For convenience of the students we briefly recapitulate the similarity of triangles.

Equiangularity : Two polygons having the same number of sides with successive equal angles are said to be equiangular.

Similarity : Two polygons having the same number of sides are said to be similar if one can establish a one-one correspondence among their vertices such that

(1) The corresponding angles are equal, and
(2) The corresponding sides are proportional.

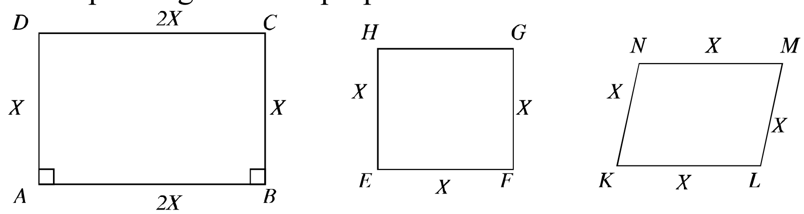


Fig : 3.10

Observing the figures above, we see that,

(1) The rectangle *ABCD* and the square *EFGH* are equiangular but not similar.

(2) The square *EFGH* and the rhombus *KLMN* are not similar, though their corresponding sides are proportional because of any successive matching of their vertex.

It is not the same of course with the case of the two triangles. If one condition of the mentioned two is the concerning definition is true due to the matching of the angles of the vertices, the other is to be true and the two triangles become similar.

In this connection it may be mentioned that,

(1) If two triangles are equiangular, each pair of equal angles are called corresponding angles and the sides opposite corresponding angles are called corresponding sides.

(2) If the three sides of a triangle are proportional to the sides of another triangle, each pair of proportional sides are called corresponding sides, and the angles opposite the corresponding sides are called corresponding angles.

(3) In both cases the triangles are described by matching a one-one correspondence among their vertices of the angles. For example, the corresponding angles are $\angle$A and $\angle$D, $\angle$B and $\angle$E, $\angle$C and $\angle$F; the example, the corresponding sides are *AB* and *DE*, *AC* and *DF*, *BC* and *EF* of $\Delta ABC \mid \Delta DEF$.

Some theorems about the similarity of the two triangle are briefly described below:

Theorem 3.6 :

If two triangles are equiangular, their corresponding sides are proportional.

In the adjacent figures, $\Delta ABC \mid \Delta DEF$ are equiangular triangles. i.e.

$\angle A = \angle D, \angle B = \angle E$

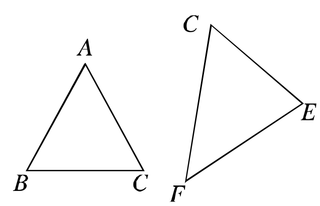


Fig : 3.11

and $\angle C = \angle F$. then $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$

That is, the corresponding sides are then proportional.

Corollary : If the two angles of one triangle are equal to the two angles of the other, the two triangle are equiangular, hence similar. Because, the sum of the three angles of any triangle is two right angles.

Theorem 3.7 :

If the sides of the two triangles are proportional, the opposite angles of the corresponding sides are mutually equal.

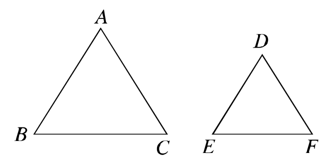


Fig : 3.12

In the adjacent figure, ΔABC | ΔDEF are such that their corresponding sides are proportional :

i.e., as $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$ The angles are mutually equal.

That is, $\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$. |

The theorem 3.7 can be called the converse proposition of the theorem 3.6

Theorem 3.8 :

If one angle of one triangle is equal to an angle of another triangle and the side adjoining the equal angles are proportional, the two triangles will be similar.

In the figure (figure 3.13) ΔABC | ΔDEF are such that $A = D$, and the sides *AB, AC* and *DE, EF* adjoining the equal angles are proportional. i.e., as $\dfrac{AB}{DE} = \dfrac{AC}{DF}$, ΔABC | ΔDEF are similar.

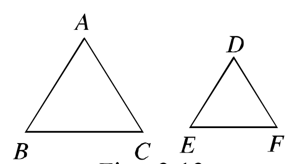


Fig : 3.13

Theorem 3.9 :

The ratio of the areas of the two similar triangles is equal to the ratio of the areas of the squares drawn of their two corresponding sides.

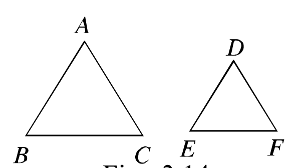


Fig : 3.14

In the adjacent figure, ΔABC | ΔDEF are similar triangles. *BC* and *EF* are the corresponding sides of the two triangles.

In this condition, the ratio of the two triangles is equal to the ratio of the squares drawn on the two sides *BC* and *EF*.

That is, $\dfrac{\Delta ABC}{\Delta DEF} = \dfrac{BC^2}{EF^2}$ |

In the case of the triangles, based on theorem and discussions the logical proof of the theorem is presented below.

Theorem 3.10 :

The circumcenter, the centroid and the orthocenter of any triangle are collinear.

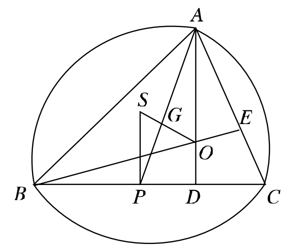


Fig : 3.15

Particular Enunciation : Suppose, *O* is the orthocenter, *S* is the circumcentre and *AP* is the median of the triangle *ABC*.

The orthocenter *O* and the circumcentre *S* intersect the median *AP* at *G*. If we join *S*, *P*, the line *SP* is a perpendicular on *BC*. So, it is enough to prove that the point *G* is the centroid of ΔABC.

We know that the distance of the Vertex from the orthocenter of any triangle is twice the distance of the opposite side of the vertex from the circumcentre of the triangle. *OA* is the distance of the vertex *A* from the orthocentre O and SP is the distance of the opposite side BC of the vertex A from the circumscentre S of ΔABC.

$OA = 2SP$ …………… (i)

Now both *AD* and *SP* are perpendicular on *BC*, *So*, $AD \parallel SP$.

Now, $AD \parallel SP$ and AP is their intersector

$\therefore \angle PAD = \angle APS$ [Alternate angle]

i.e., $\angle OAG = \angle SPG$

Now in between ΔAGO and ΔPGS

$\angle AGO = \angle PGS$ [Vertical opposite angle]

$\angle OAG = \angle SPG$ [Vertical angle]

$\therefore$ remaining $\angle$AOG = remaining $\angle PSG$

$\therefore$ ΔAGO and ΔPGS are equiangular.

So, $\dfrac{AG}{GP} = \dfrac{OA}{SP}$

Or, $\dfrac{AG}{GP} = \dfrac{OA}{SP}$

Or, $\dfrac{AG}{GP} = \dfrac{2SP}{SP}$ [from the equation (1)]

Or, $\dfrac{AG}{GP} = \dfrac{2}{1}$

$\therefore\ AG : GP = 2:1$

i.e., The point G divies the median AP into the ratio $2:1$

$\therefore\ G$ is the centroid of ΔABC (proved)

N.B.

(1) Nine Point Circle :

The total nine points including the tri-middle points of the sides, feet of the three perpendiculars drawn from each vertex to the opposite side and the middle points of the three line segments joining the orthocenter to the vertices of any triangle lie on one circle. This circle is called the nine point circle.

(3) The centre of the nine point circle is the middle point of the line segment joining the orthocenter and the circumcentre.

(4) The radius of the nine point circle is half of the circumradius.

Theorem 3.11 (The theorem of Brahmagupta) :

In any triangle the area of the rectangle included by any two sides is equal to the area of the rectangle included by the diameter of the circumcircle and the perpendicular drawn from the initial point of the two sides on the opposite side.

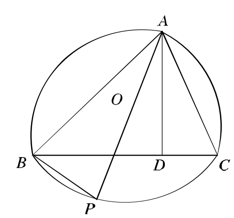


Fig : 3.16

Particular Enunciation : Suppose, O is the circumcentre and AP is a diameter of the circumcircles of the triangle ABC. AD is the perpendicular drawn from the vertex A opposite side BC. It is to be proved that $AB.\ AC = AP.\ AD$

Construction : Join B and P

For the same arc AB, $\angle APB$ and $\angle ACD$ are the angles in the segment of a circle. AS, AP is the diameter, ABP is a semi-circled angle and as AD is a perpendicular on the side BC, $\angle ADC$ is a right angle.

Now in between ΔAPB and ΔACD, $\angle APB = \angle ACD$ [the angles on the same segment of circle are equal.]

$\angle ABP$ = semi-circled angle = 1 right angle = $\angle ADC$.

$\therefore$ remaining $\angle BAP$ = remaining $\angle CAD$.

$\therefore\ \Delta ABP$ ∣ ΔADC |

$\therefore \dfrac{AB}{AD} = \dfrac{AP}{AC}$

$\therefore AB \cdot AC = AP \cdot AD.$ [proved]

Remark : If R is the circumradius of the triangle ABC ; then $AP = 2R$; so $AB \cdot AC = 2R \cdot AD$.

Brhmupta was a prominent Indian Mathematician who flourished in the early decades of the seventh century of the Christian era.

Our last theorem is due Ptolemy, the great astronomer and mathematician of the second century of the Christian era.

Theorem 1.12 : (Ptolemy's Theorem) : In any cyclic quadrilateral the area of the rectangle contained by the two diagonal is equal to the sum of the area of the two rectangles contained by the two pairs of opposite sides.

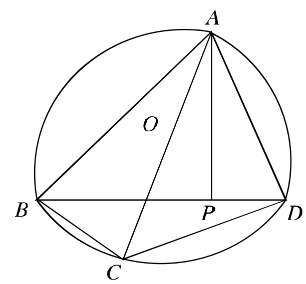


Fig : 3.17

Particular Enunciation : Suppose $ABCD$ is a cyclic quadrilateral ; AC and BD are its diagonals and AB, CD and BC, AD are its two pairs of opposite sides.

The prove $AC \cdot BD = AB \cdot CD + BC \cdot AD$.

Construction : Without loss of generality we can assume that $\angle BAC$ is smaller than $\angle DAC$; we draw $\angle DAP$ making it equal to $\angle BAC$, as shown in figure 3.17.

Proof : We have $\angle BAC = \angle DAP$ (by construction).

Adding $\angle CAP$ to both sides we get

$\angle BAP = \angle CAD$ because $\angle BAP = \angle BAC + \angle CAP$ and $\angle CAD = \angle DAP + \angle CAP$.

$\therefore \angle ABD = \angle ACD$ (angles on the same segment $ADCP$ of the circle)

So, in the triangles ABP and ACD besides $\angle BAP = \angle CAD$, we also have

$\therefore$ remaining $\angle APB$ = remaining $\angle ADC$.

So, ΔABP and ΔACD are equiangular, hence similar.

Therefore $\dfrac{BP}{CD} = \dfrac{AB}{AC} \Rightarrow AC \cdot BP = AB \cdot CD$ ………….. (1)

Again, in the triangles ABC and APD we have $\angle BAC = \angle DAP$ (construction) and $\angle ADP = \angle ACB$, because $\angle ADP = \angle ACB$ and because $\angle ADP$ and $\angle ACB$ are equal, being angles on the same segment $APBC$ of the circle.

$\therefore$ remaining $\angle ABC$ = remaining $\angle APD$.

So, the triangle ABC and ADP are equiangular, hence similar.

$\therefore \dfrac{AD}{AC} = \dfrac{PD}{BC} \Rightarrow AC \cdot PD = BC \cdot AD$ ……….. (2)

Adding the relation (1) and (2) we get,

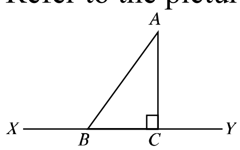
$AC \cdot BP + AC \cdot PD = AB \cdot CD + BC \cdot AD$

or, $AC(BP + PD) = AB \cdot CD + BC \cdot AD$

so, $AC \cdot BD = AB \cdot CD + BC \cdot AD$ [as, $BP + PD = BD$] [Proved]

Exercise 3.2

1. Refer to the picture below and answer the question :

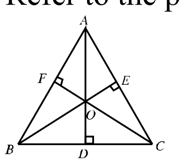


Which is the orthogonal projection of the line segment AB on XY ?

a. AB b. BC

c. AC d. XY

2. Refer to the picture below land answer the question :



Which point is the orthocentre of the triangle ABC ?

a. D b. E

c. F d. 0

3. Study the following statements :

i. The point of concurrence of the medians of a triangle is called the centred

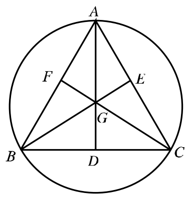
ii. The centroid divides any median in the ration 3 : 1.

iii. The corresponding angles of two equiangular triangles are proportional.

Which combination of these statements is correct ?

a. i and ii b. ii and iii

c. i and iii d. i, ii and iii



In the picture below *D, E, F* are respectively the middle points of *BC, AC*, and *AB* ; in the light of this picture answer questions 4, 5 and 6.

4. What is the name of the point G ?

   a. orthocentre b. Incentre

   c. Cetroid d. Circumcentre

5. What is the name of the circle drawn through the vertices of ABC ?

   a. Circumcircle b. Incircle

   c. Excircle d. Nine Point Circle

6. Which statement is consistent with the Theorem of Apollonius applied to ABC ?

   a. $AB+AC^2=BC^2$ b. $AB^2+AC^2=2(AD^2+BD^2)$

   c. $AB^2+AC^2=2(AG^2+GD^2)$ d. $AB^2+AC^2=2(BD^2+CD^2)$

7. From any point p lying on the circumcircle of the triangle perpendiculars PD, PE are drawn on BC, CA respectively. The line segment ED intersects at the point 0 ; prove that $PO \perp AB$ (PO is perpendicular to AB).

8. C in ΔABC is a right angle, and CD is the perpendicular drawn from the vertex C on the hypotenuse. Prove that $CD^2=AD\cdot BD$.

9. 0 be the orthocentre of ABC ; prove that $AO\cdot OD=BO\cdot OE=OC\cdot OF$ D,E,F be the foot of the perpendicular from the vertex A,B,C on the opposite sides.

10. A semicircle is drawn with diameter AB ; two of its chords AC and BD intersect at P. Prove that $AB=AC\cdot AP+BD\cdot BP$.

11. The circumradius of an equilateral triangle is 3.0 cm ; find the length of a side of the triangle.

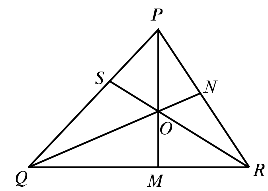
    Hint : BCF and acoe are similar triangles ; so $BO:CO=OP:OE$

12. In the isosceles triangle ABC, AD is the perpendicular from vertex A to the base. Prove that $AB^2=2R\cdot AD$ Where R is the circumradius of the triangle.

    Hint : D is the middle point of BC .

13. The bisector of the angle A of the triangle ABC intersects BC at D and the circumcircle at E. Show that $AD=AB\cdot AC-BD\cdot DC$.

14. In the triangle ABC, BE and CF are perpendiculars on the sides AC and AB, respectitively. Show that $\Delta ABC : \Delta AEF = AB^2 : AB^2$

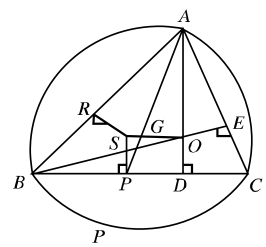


15. If ΔPQR the medians PM, QN and RS pass through the point G.

a. What is the name of the point G ? In what ratio does G divide PM ?

b. Establish the relation $PQ^2 + PR^2 = 2\,(PM^2 + QM^2)$. By what name is this relation known ?

c. Show that the sum of squares of the three sides of PQR is equal to four times the sum of squares of the distances of the three vertices from G.



16. In the picture below S is the circumcentre and 0 is the orthocentre of the

of the triangle $ABC; AP$ is the median from A, a = BC, b = AC and AB = c .

a. Establish a relation between OA and SP.

b. If C is an acute angle, show that a. CD = b. CE.

Chapter Four

Geometric Constructions

The figure drawn by using compass and ruler according to the definite condition is geometric construction. Geometric figures drawn for proving the theorems need not be accurate. But in geometric construction, figure needs to be accurate.

After completing the chapter, the students will be able to –

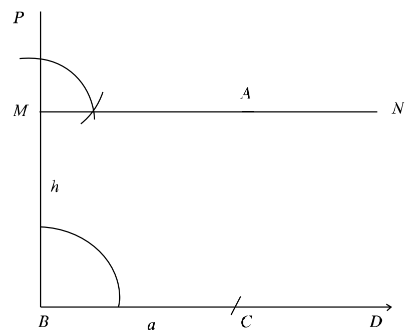
- Construct the triangles on the basis of given data and information and justify construction.
- Construct the circles on the basis of given data and information and justify construction.

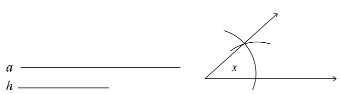
4.1 Some constructions Involving Triangles

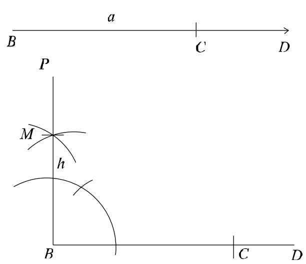
Construction 1 :

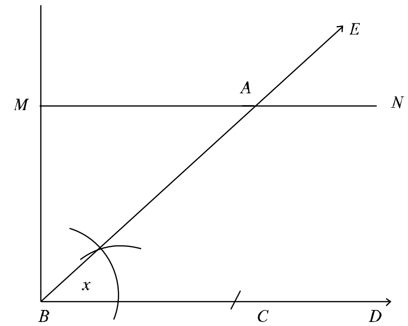
The base, an angle adjoining the base and height the triangle are given. The triangle needs to be drawn.

Suppose, the base a, the height h and an angle x adjoining the base are given. The triangle needs to be drawn.









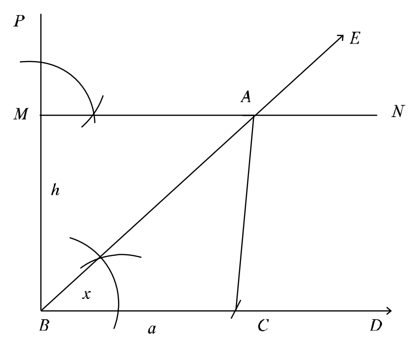
Description of the Construction :

Step 1 : Cut the part $BC = a$ from any ray BD.

Step 2 : Draw at B the perpendicular BP on the base BC.

cut $BM = h$ from BP.

Step 3 : Draw through the point M, the line MN parallel to BC.



Step 4 : Again construct $\angle CBE$, equal to the given $\angle x$. The line segment BE intersects MN at A.

Step 5 : Join A and C. Then ABC is the desired triangle.

Proof : Since $MN \parallel BC$ (As per the construction)

$\therefore$ The height of ΔABC is $BM = h$

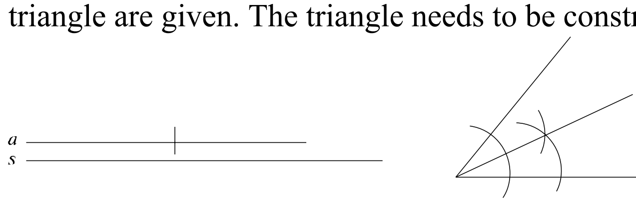
Again $BC = a$ and $\angle ABC = \angle x$

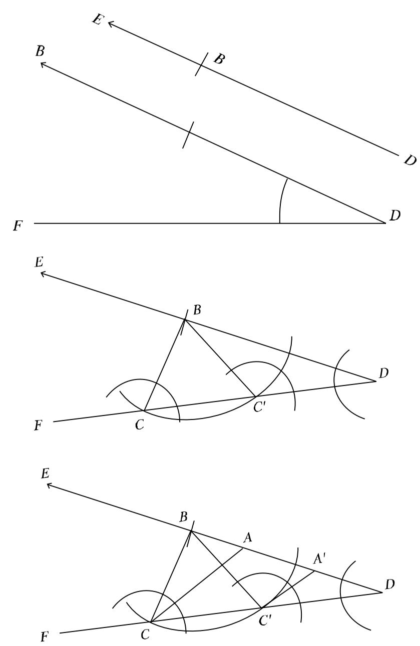
$\therefore$ ΔABC is the desired triangle.

Analysis : The base and an angle adjoining the base are given. So we need to cut off a portion from a ray equal to the base. We draw the perpendicular at one end point of the base and cut off a portion equal to the height. Finally, at that end- point we draw an angle equal to the given angle. The point where the other arm of this angle intersects the line parallel to the base as the given height, is the third vertex of the desired triangle.

Construction 2 :

The base, the vertical angle and the sum of the lengths of the other two sides of a triangle are given. The triangle needs to be constructed.





Let ‘a’ be the base, ‘ x ’ be the vertical angle, s be the sum of the other two sides. The triangle needs to be constructed.

Description of the Construction :

Step 1 : Cut the segment $DB = S$ from any ray DE.

Step 2 : At D of the line BD, draw $\angle BDF = \frac{1}{2}\angle x$.

Step 3 : Taking B as centre, draw a segment of circle of radius a; let it intersect DF at C and C'. Join B, C and B, C'.

Step 4 : At the point C, draw $\angle BDF$ equal to $\angle DCA$ and at C, draw $\angle CD'A'$ equal to $\angle BDF$. Let CA and $C'A'$ intersect BD at A and A' respectively.

Then both the triangles ABC and $A'BC'$ are the required triangle.

Proof : Since $\angle ACD = \angle ADC = \angle A'C'D = \frac{1}{2}x$ (by construction)

$$\therefore \quad \angle BAC = \angle ADC + \angle ACD = \frac{1}{2}\angle x + \frac{1}{2}\angle x = \angle x$$

$$\angle BA'C' = \angle A'DC' + \angle A'CD' = \frac{1}{2}\angle x + \frac{1}{2}\angle x = \angle x$$

And $AC = AD$, $A'C' = A'D$.

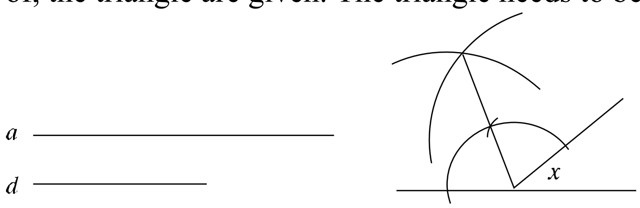
So, in the triangle $ABC, \angle BAC = \angle x, BC = a$ and $CA + AB = DA + AB = DB = s$;

And $A'BC'', \angle BA'C' = \angle x,\ BC' = a$ and $C'A' = A'B = DA' + A'B = DB = s$.

$\therefore\ \Delta A'BC'$, is the other required triangle.

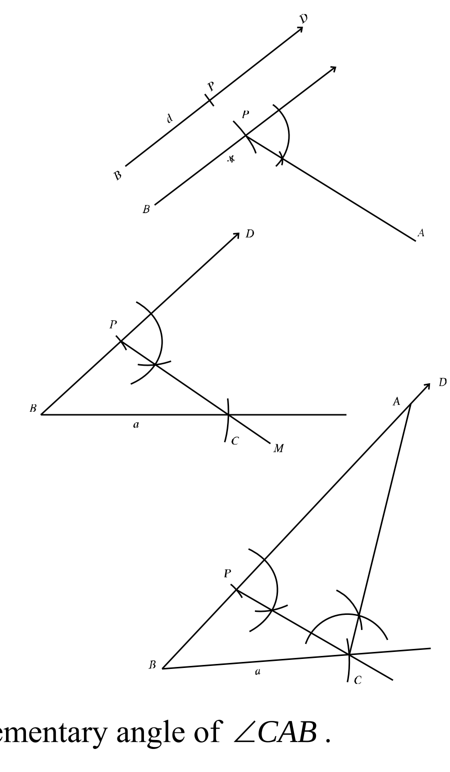
Construction 3 :

The base, the vertical angle and the difference of the lengths of the other two sides of, the triangle are given. The triangle needs to be constructed.



Let a be the base. Given that d is the difference of the other two sides and x is the vertical angle.

Description of the Construction :



Step 1 :

Cut the segment $BP = d$ from any ray BD.

Step 2 : At P, draw $\angle DPM$, equal to the half of the supplementary angle of $\angle x$.

Step 3 : Taking B as centre, all are with the radius a of the circle ; let the arc equal to the radius intersect the straight line PM at the point C .

Step 4 : Join B and C .

Step 5 : Again, at the point C draw, $\angle DPC = \angle PCA$.

Let the line segment CA intersect BD at A .

Then ABC is the required triangle.,

Proof : $\angle APC = \angle ACP \quad \therefore \quad AP = AC$

$\therefore \quad AB - AC = AB - AP = d$.

Again, $\angle APC = \angle ACP$ = half of the supplementary angle of $\angle x$.

$\therefore \quad \angle APC + \angle ACP$ = Supplementary of $\angle x$.

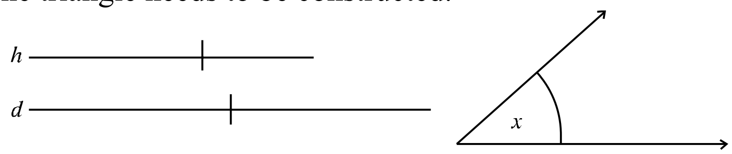
= external $\angle CAD$ = supplementary angle of $\angle CAB$.

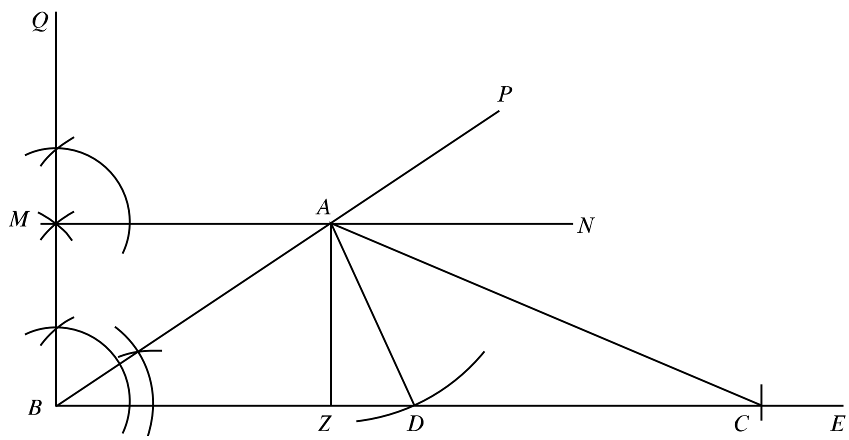
$\therefore \angle A = \angle CAB = \angle x$.

$\therefore$ ABC is the required triangle.

Construction 4 :

The height, the median on the base and an angle adjoining to the base of the triangle are given. The triangle needs to be constructed.





Let h be the height, d be the median on the base, x be an angle adjoining to the base of the triangle. The triangle needs to be constructed.

Description of the Construction :

Step 1 : Draw BE and draw $\angle EBP$ equal to $\angle x$ at B of any ray BE.

Step 2 : At the point B, draw BQ perpendicular on the line BE.

Step 3 : From BQ, cut BM equal to the height h of the triangle.

Step 4 : At the point M, draw the line NM parallel to BE which intersects BP at the point A.

Step 5 : Taking A as centre, draw an arc with the radius equal to the median d; let the arc intersect BE at the points D.

Step 6 : From BE, cut the segment $BD = CD$.

Step 7 : Join A, C.

Then ABC **is the required triangle**

Proof : Join A, D ; draw AZ perpendicular from A on BC.

Here, MN and BE are parallel line and both MB and AZ are the perpendiculars on BE.

$\therefore$ $MB = AZ = h$ = height.

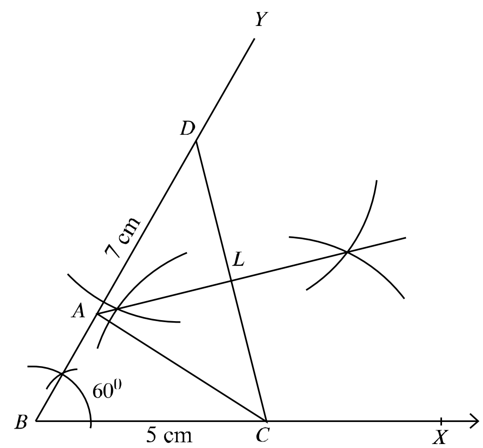
$\therefore$ $BD = DC$, the point D is the middle point of BC, $AD = d$ = the median drawn on the base, i.e., the base BC.

$\angle ABC = \angle x$ = angle = adjacent to the given length of the median.

$\therefore$ Again, ABC is the required triangle.

Remark : We can get two triangle in many cases depending on $\angle x$.

Example 1. Given that the length of the base of a triangle is 5 cm and angle adjoining the base is 60^{o} and the sum of the lengths of the two other sides is 7 cm. Construct the triangle.



Solution : It is given that $BC = 5$ cm. $AB + AC$ = 7 cm $\angle ABC = 60^{\circ}$. Construct ΔABC..

Steps of the construction :

Step 1 : From any ray BX, cut off BC = 5cm

Step 2 : Draw $\angle XBY = 60^{\circ}$

Step 3 : From the ray BY, cut BD = 7 cm

Step 4 : Join C, D.

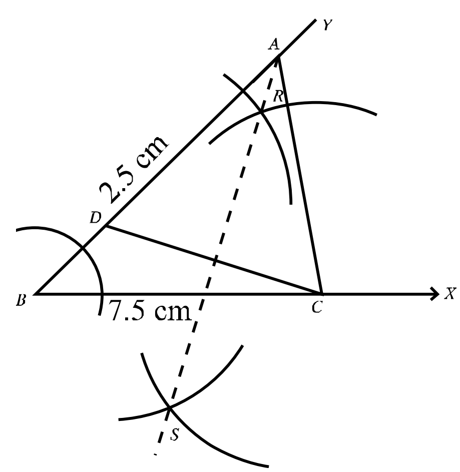
Step 5 : Draw the perpendicular bisector of CD; let it intersect BD at the point A.

Step 6 : Join A, C. Then ABC is the required triangle. AL is the perpendicular bisector of CD

$\therefore$ $AD = AC$

Then $BD = BA + AD = BA + AC$ = 7 cm.

Example 2. The length of the base of a triangle is 7.5 cm an angle adjoining the base is 45^{o} and the difference of the lengths of the other two sides is 2.5 cm. Construct the triangle.



Solution : Given that the base = 7.5 cm, the difference of the other two sides $AB - AC$ or $AC - AB$ = 3.5 cm and the angle adjoining the base 45^{o}. Construct the triangle.

Steps in the Construction of $AB - AC$ = 2.5 cm:

1. From any ray BX cut $BC = 7.5$
2. Draw $\angle YBC = 45^o$
3. From the ray BY, cut $BD = 2.5$cm.
4. Join C, D.
5. Draw the perpendicular bisector RS of CD ; let it intersect BY at the point A.
6. Join A, C.

Then ABC is the desired triangle.

(ii) The students are advised to draw the required triangle in the case $AC - AB = 2.5$ cm.

Activity :

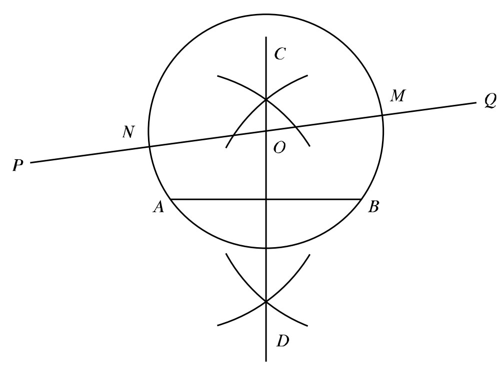
1. The perimeter and the two angles adjoining the base of a triangle are given. Draw the triangle.
2. The base $BC = 4.6$ cm, $\angle B = 45^o$ and $AB + CA = 8.2$ cm. of the triangle are given. Draw the triangle.
3. In a right angled triangle, one side has length 3.5cm, the sum of the lengths of the other side and the hypotenuse is 5.5 cm. are given. Draw the triangle.
4. The base $BC = 4.5$ cm, $B = 45^o$ and $AB - AC = 2.5$ cm of a triangle are given. Draw the triangle ABC.
5. The perimeter of ΔABC is 12 cm, $\angle B = 60^o$ and $\angle C = 45^o$ of ΔABC are given. Draw ΔABC.

4.2 Some Constructions Involving Circles

Constructions 5 :

Draw such a circle which passes through two definite points and whose centre lies on a definite straight line.

A and B are the two fixed points, PQ is a fixed straight line. Construct such a circle which passes through the points A and B and whose centre lies on the straight line PQ.



Steps of the Construction :

Step 1 : Join A and B.

Step 2 : Construct the perpendicular bisector CD of line segment AB.

Step 3 : The line segment CD intersects the line PQ at the point O.

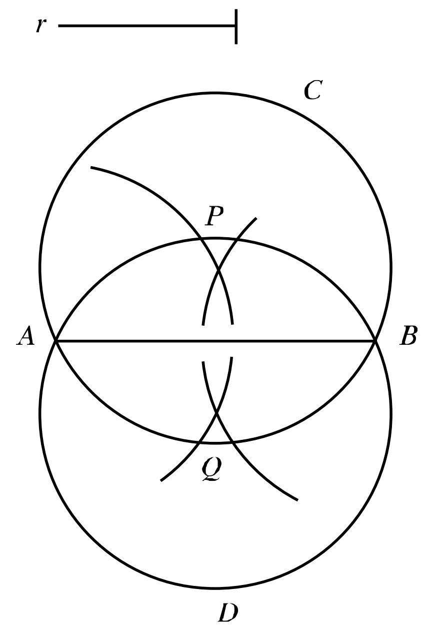
Step 4 : Taking O as centre draw the circle of radius OA or OB, $ABMN$ is the desired circle.

Proof : $OA = OB$ because O lies on the perpendicular bisector of the line segment AB. The centre of the circle $ABMN$. Which is 0, lies on PQ, and the circle passes through the points A and B. So $ABMN$ is the required circle.

Construction 6 :

With the radius equal to a definite line segment, construct a circle which passes through two definite points.

A and B are the definite points and r is the length of the definite line segments. Construct such a circle which passes through A & B and whose radius is equal to r.



Steps of the Construction :

1 : Join A and B.

2 : Draw two segments of the two circles of radius and centre A and B, on either side of the line AB, The two pairs of segments of the two circles intersect at P and Q on the two sides of the line AB.

3 : Taking P as centre and PA as radius, draw the circle ABC.

4 : Taking Q as centre and QA as radius, draw the circle ABD. Then each of ABC and ABD is the required circle.

Proof : $PA = PB = r$

$\therefore$ draw the circle ABC with centre P and radius PA or PB, passes through the points A and B, and its radius is $PA = r$.

Again, $QA = QB = r$.

The drawn circle ABC with centre Q and radius QA or $QB = r$ passes through the point A and B, and its radius is $QA = r$.

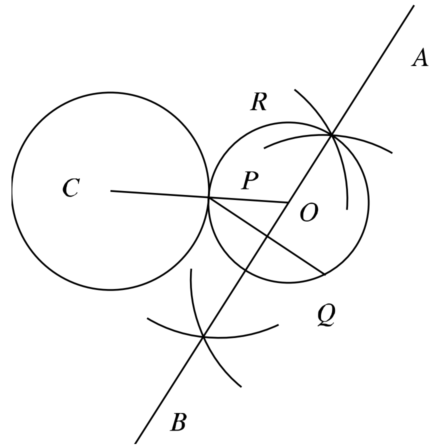
So, each of the two circle ABC and ABD, is the required circle.

Construction 7 :

Construct a circle which touches a definite point of a definite circle and passes through a definite point outside the circle.

Let a circle be given with centre O, P be a definite point on that circle and Q be a definite point outside that circle. Draw a circle which touches the circle at P and passes through the point Q.

Steps of the Construction :



Step 1 : Join P and Q.

Step 2 : Draw the perpendicular bisector AB of PQ.

Step 3 : Join C, P.

Step 4 : Extended line segment CP intersects AB at the point O.

Step 5 : Taking O as centre, draw the circle with radius equal to OP. The resulting circle PQR is the required circle.

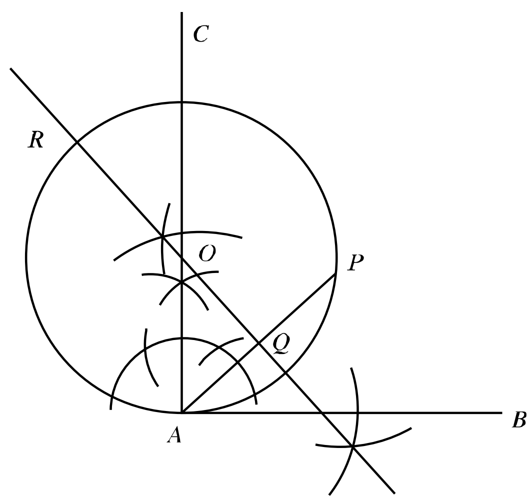
Proof : Join O, Q. The line segment AB or the line segment OB is the bisector of PQ.

$\therefore OP = OQ.$

So the circle of radius = OP and centre O will pass through Q. The point P lies on the given circle and on the constructed circle and also the line joining the centres of the two circles. So the two circles touch at P. So the circle is the desired circle.

Construction 8 :

Construct a circle which touches a definite point on a definite straight line and passes through a point.



Suppose A is a definite point on the straight line AB and P be a point not lying on the line AB. To draw a circle which touches the line AB at A and passes through the point P.

Steps of the Construction :

Step 1 : Draw the perpendicular AC at the point A on the line AB.

Step 2 : Join A, P and construct its perpendicular bisector QO.

Step 3 : The lines QO and AC intersect at O.

Step 4 : Taking O as centre draw the circle with radius OA, which intersects the extended line QO at R.

Then APR is the required circle.

Proof : Join O, P.

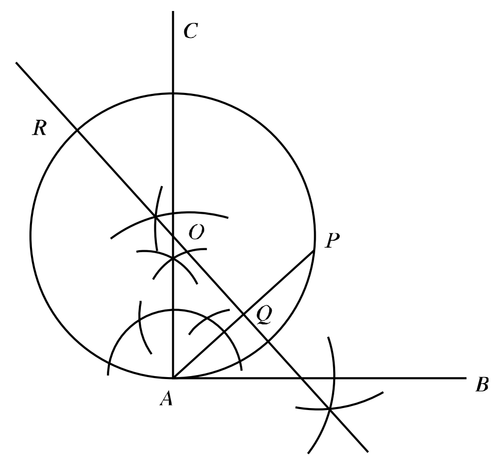
$\therefore$ $OA = OP$. The point O lies on the perpendicular bisector of AP.

$\therefore$ The circle with centre O and radius OA passes through the points P.

Again, OA passing through A is a perpendicular to the line AB ; so the circle touches the line AB at the point A.

$\therefore$ Taking O as centre and OA as radius, the drawn circle is required circle.

Analysis : The circle is required to touch a definite line at a definite point; so that line has to be tangent to the circle at that definite point. So the diameter of the circle passing through the definite point will be perpendicular to the definite line. Since the definite point on the line and the definite external point both are required to lie on the circle, the perpendicular bisector of the line segment joining the two points, will pass through the centre. So the centre is the point of intersection of this bisector with the perpendicular erected at the point given on the definite line.



Example 1. Given a point at a distance of 5cm from the centre of a circle of radius 2 cm. Determine the distance of the two tangents.

Solution : Take a point 0, draw a circle with centre at 0 and radius 2 cm. Fix a point P at a distance of 5 cm from 0.

We need to construct the two tangents to the circle for determining their lengths.

Steps of the Construction :

Step 1 : Bisector line OP. Let M be the bisector point.

Step 2 : Draw the circle with centre at M and radius OM ; it intersects the circle at the points Q and R.

Step 3 : Join P, Q and P, R.

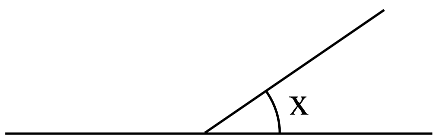
Then PQ and PR are the two required tangents. Measuring PQ and PR we find $PQ = PR = 4 \cdot 6$ cm.

Activity :

1. By drawing the incircle of a triangle whose sides have lengths 5 cm., 12 cm. and 13 cm., measure the length of its radius.
2. By drawing the circumcicle of a triangle whose sides have lengths 6.5 cm., 7 cm. and 7.5 cm., measure the length of its radius.

Exercise 4

1. If $x = 60$, what is the measurement of the half of the supplementary angle of $\angle x$?



(a) 30° (b) 60° (c) 120° (d) 180°

2. (i) A triangle cannot be drawn if the lengths of three sides are given arbitrarily.

   (ii) A circle can be drawn if only the radius is known.

   (iii) At any point on a circle only one tangent can be drawn.

Which one of these statements is correct ?

(a) i and ii (b) ii and iii (c) i and iii (d) i, ii and iii

3. Two angles and the difference of the lengths of their opposite sides of any triangle are given, draw the triangle.
4. The base, the difference of the angles adjoining the base and the sum of the other two sides are given, draw the triangle.
5. The base, the vertical angle and the sum of the other two angles are given. Draw the triangle.
6. The base, the vertical angle and the difference of the other two angles are given. Draw the triangle.
7. The length of the hypotenuse and the sum of the other two sides of a right angle triangle are given. Draw the triangle.
8. An angle adjoining the base, the height and the sum of the other two sides, draw the triangle.
9. Given the length of the hypotenuse and the difference of the lengths of the other two sides are given. Draw the triangle.

10. Draw a circle which a definite straight line at a definite point and another circle touches.
11. Draw a circle which touches a definite straight line at a definite point and the another circle at any point.
12. Draw a circle which touches a given straight line at some point and also touches a given circle at a point given on it.
13. Draw three circle of such different radius that they touch each other externally.
14. P is any point on a chord AB of a circle. Draw a chord through P such that $CP^2 = AP.\ OB$.
15. In an isosceles triangle, the equal sides have length 6 cm. the base has length 5 cm.

a. Draw the triangle.

b. Draw the incircle of the triangle and measure its radius.

c. Draw separately the circumcircle of the triangle and measure its radius.

d. Draw a circle which touches a point P whose radius is equal to the circumradius of above triangle and which passes through the point Q outside that circle.

Chapter Five

Equations

It has been before that unknown quantities or variables are very important in Algebra. In practical life, we use *x*, *y*, *z* etc. as symbols to represent unspecified object, number or objects. These symbols are called variables or unknown quantities. An expression is formed with more than one unknown quantitiy or variable; such as $2x + y$, $x^2 + z$, $x + y + 2z$ etc. When an unknown quantity or expression is equated to a definite number or value, then it is called an equation. Equation is a very important topic in Algebra. A lot of practical problems can be solved using equations.

At the end of this chapter, the students will be able to

- solve quadratic equations $(ax^2 + bx + c = 0)$.
- identify the equations involving square roots
- to solve the equations involving square roots
- to explain indicial equations
- to solve indicial equations
- to solve system of linear and quadratic equations of two variables
- to express practical problems in linear and quadratic equations of two variables and solve them
- to solve system of indical equations of two variables
- to solve quadratic equations $(ax^2 + bx + c = 0)$ graphically.

5.1 Quadratic equations of one variable and their solutions

Linear and quadratic equations in one variable and linear equations in two variables have been discussed in details in Secondary Algebra Book. A quadratic equation in one variable can be solved easily after factorizing the left hand side of the equation if the roots are rational. But all expressions cannot be factorized easily. That is why the following procedure is used to solve the quadratic equation of any form.

The standard form of quadratic equations in one variable is $ax^2 + bx + c = 0$. Here *a*, *b*, *c* are real numbers and *a* can never be zero.

Let us solve the quadratic equation

$ax^2 + bx + c = 0$

or, $a^2x^2 + abx + ac = 0$ [multiplying both sides by a]

or, $(ax)^2 + 2(ax)\dfrac{b}{2} + \left(\dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2 + ac = 0$ or, $\left(ax + \dfrac{b}{2}\right)^2 = \dfrac{b^2}{4} - ac$

or, $\left(ax+\frac{b}{2}\right)^2=\frac{b^2-4ac}{4}$ or, $ax+\frac{b}{2}=\pm\frac{\sqrt{b^2-4ac}}{2}$

or, $ax=-\frac{b}{2}\pm\frac{\sqrt{b^2-4ac}}{2}$ or, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ (*i*)

Therefore, the two values of x are

$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$ (*ii*) and $x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$ (*iii*)

In equation (i) above, $b^2 - 4ac$ is called the Discriminant of the quadratic equation because it determines the state and nature of the roots of the equation.

Variations and nature of the roots of a quadratic equation depending on the conditions of the dicriminant.

(i) If $b^2 - 4ac > 0$ and is a perfect square, then the two roots of the equation are real, unequal and rational.

(ii) If $b^2 - 4ac > 0$ but is not a perfect square, then the two roots of the equation are real, unequal and irrational.

(iii) If $b^2 - 4ac = 0$, then the roots of the equation are real and equal. In this case, $x=-\frac{b}{2a}, -\frac{b}{2a}$.

(iv) If $b^2 - 4ac < 0$, then the roots of the equation are not real. In this case, the two roots are always conjugate complex or imaginary to each other. You will learn this in higher class.

Example 1. Solve $x^2-5x+6=0$.

Solution: Comparing the given equation with $ax^2+bx+c=0$, we get $a=1, b=-5$ and $c=6$. So the solutions of the equation are

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4.1.6}}{2.1}=\frac{5\pm\sqrt{25-24}}{2}=\frac{5\pm\sqrt{1}}{2}$$

$$=\frac{5\pm1}{2}=\frac{5+1}{2}, \frac{5-1}{2}$$

That is, $x_1=3, x_2=2$.

Example 2. Solve $x^2-6x+9=0$.

Solution: A comparision with $ax^2+bx+c=0$ gives $a=1, b=-6$ and $c=9$. Hence, we obtain

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4.1.9}}{2.1}=\frac{6\pm\sqrt{36-36}}{2}=\frac{6\pm0}{2}$$

i.e. $x_1 = 3,\ x_2 = 3$.

Example 3. Solve: $x^2 - 2x - 2 = 0$.

Solution: After comparing this equation with the standard quadratic equation we can write $a = 1,\ b = -2,\ c = -2$.

Therefore, the two roots are

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4.1.(-2)}}{2.1} = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2}$$

or, $x = \dfrac{2 \pm 2\sqrt{3}}{2} = \dfrac{2(1 \pm \sqrt{3})}{2}$

i.e., $x_1 = 1 + \sqrt{3},\ x_2 = 1 - \sqrt{3}$.

It is noticed here that though $x^2 - 2x - 1$ cannot be factorized with rational numbers, yet it has been possible to solve the equation by this method.

Example 4. Solve: $3 - 4x - x^2 = 0$

Solution: Comparing the equation with the standard quadratic equation $ax^2 + bx + c = 0$, we get $a = -1,\ b = -4,\ c = 3$.

$\therefore$ The two roots are

$$x = \frac{-(-1) \pm \sqrt{(-4)^2 - 4.(-1).3}}{2.(-1)} = \frac{1 \pm \sqrt{16+12}}{-2} = \frac{4 \pm \sqrt{28}}{-2} = \frac{4 \pm 2\sqrt{7}}{-2}$$

or, $x = -(2 \pm \sqrt{7})$

i.e., $x_1 = -2 - \sqrt{7},\ x_2 = -2 + \sqrt{7}$.

> **Activity**: Use the formulae (ii) and (iii) mentioned above to find the values of x_1 and x_2 from $ax^2 + bx + c = 0$, when $(i)\ b = 0,\ (ii)\ c = 0\ \ (iii)\ b = c = 0$ $(iv)\ a = 1$ and $(v)\ a = 1,\ b = c = 2p$

Exercise 5.1

Use formula to solve the following equations:

1. $2x^2 + 9x + 9 = 0$	2. $3 - 4x - 2x^2 = 0$	3. $4x - 1 - x^2 = 0$
4. $2x^2 - 5x - 1 = 0$	5. $3x^2 + 7x + 1 = 0$	6. $2 - 3x^2 + 9x = 0$
7. $x^2 - 8x + 16 = 0$	8. $2x^2 + 7x - 1 = 0$	9. $7x - 2 - 3x^2 = 0$

6.1 Equations involving radicals.

We know that the value or values of a variable which make both sides of an equation equal are the roots of the equation and it is satisfied by that value or those values.

In an equation, the quantities with a variable involving a square root sign are freed from the square root sign by squaring and a new equation is obtained. All the roots of the equation thus obtained do not satisfy the given equation. These are extraneous roots. Therefore, the roots of the equation involving the radical sign are to be tested to know whether they satisfy the given equation or not. After test, those who satisfy the given equation are the roots of the equation. Some examples are given below:

Activity: Taking $p=\sqrt{\frac{x}{x+16}}$, solve $\sqrt{\frac{x}{x+16}}+\sqrt{\frac{x+16}{x}}=\frac{25}{12}$ and then verify the result.

Example 1. Solve: $\sqrt{8x+9}-\sqrt{2x+15}=\sqrt{2x-6}$

Solution: $\sqrt{8x+9}-\sqrt{2x+15}=\sqrt{2x-6}$

or, $\sqrt{2x+15}+\sqrt{2x-6}=\sqrt{8x+9}$

or, $2x+15+2x-6+2\sqrt{2x+15}\sqrt{2x-6}=8x+9$ [squaring]

or, $\sqrt{2x+15}\sqrt{2x-6}=2x$

or, $(2x+15)(2x-6)=4x^2$ [squaring again]

or, $4x^2+18x-90=4x^2$

or, $18x=90$

$\therefore x=5$

Verification: For $x=5$, left-hand side $=\sqrt{49}-\sqrt{25}=7-5=2$ and right-hand side $=\sqrt{4}=2$.

$\therefore$ Required solution, $x=5$.

Example 2. Solve: $\sqrt{2x+8}-2\sqrt{x+5}+2=0$

Solution: $\sqrt{2x+8}=2\sqrt{x+5}-2$

or, $2x+8=4(x+5)+4-8\sqrt{x+5}$ [squaring]

or, $8\sqrt{x+5}=4x+20+4-2x-8$ [transposing]

or, $8\sqrt{x+5}=2x+16=2(x+8)$

or, $4\sqrt{x+5}=x+8$

or, $16(x+5)=x^2+16x+64$ [squaring again]

or, $16=x^2$

$\therefore x=\pm\sqrt{16}=\pm 4$

Verification: For $x=4$, left-hand side = $\sqrt{16}-2\sqrt{9}+2=4-2\times 3+2=0$ = right-hand side.

For $x=-4$, left-hand side = $\sqrt{-8+8}-2\sqrt{-4+5}+2=0-2\times 1+2=0$ = right-hand side.

$\therefore$ Required solution, $x = 4, -4$.

Example 3. Solve: $\sqrt{2x+9}-\sqrt{x-4}=\sqrt{x+1}$

Solution: $\sqrt{2x+9}-\sqrt{x-4}=\sqrt{x+1}$

or, $2x+9+x-4-2\sqrt{(2x+9)(x+4)}=x+1$ [squaring]

or, $2\sqrt{2x^2+x-36}=2x+4$

or, $\sqrt{2x^2+x-36}=x+2$

or, $2x^2+x-36=x^2+4x+4$ [squaring again]

or, $x^2-3x-40=0$

or, $(x-8)(x+5)=0$

$\therefore x=8$ or, -5

Verification: For $x=8$, left-hand side = $5-2=3$ and right-hand side = 3

$\therefore x=8$ is the root of the given equation.

$x=-5$ is not acceptable, because putting $x=-5$, each term becomes square root of a negative number which is not defined.

$\therefore$ Required solution, $x=8$.

Example 4. Solve: $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}$

Solution: $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}$

or, $\sqrt{x^2-3x+2}-\sqrt{2}=-\sqrt{x^2-7x+12}$

or, $x^2-3x+2-2\sqrt{2}\sqrt{x^2-3x+2}+2=x^2-7x+12$ [squaring]

or, $\sqrt{2x^2-6x+4}=2x-4$

or, $2x^2-6x+4=(2x-4)^2=4x^2-16x+16$ [squaring again]

or, $x^2-5x+6=0$

or, $(x-2)(x-3)=0$

$\therefore\ x=2$ or, $x=3$.

Verification: For $x=2$, left-hand side = $\sqrt{2}$ = right-hand side.

For $x=3$, left-hand side = $\sqrt{2}=$ right-hand side.

$\therefore$ Required solution, $x=2, 3$

Example 5. Solve: $\sqrt{x^2-6x+15}-\sqrt{x^2-6x+13}=\sqrt{10}-\sqrt{8}$

Solution: $\sqrt{x^2-6x+15}-\sqrt{x^2-6x+13}=\sqrt{10}-\sqrt{8}$

Now, writing $x^2-6x+13=y$, we get the equation as

$\sqrt{y+2}-\sqrt{y}=\sqrt{10}-\sqrt{8}$

or, $\sqrt{y+2}+\sqrt{8}=\sqrt{y}+\sqrt{10}$

or, $y+2+8+2\sqrt{8y+16}=y+10+2\sqrt{10y}$ [squaring]

or, $\sqrt{8y+16}=\sqrt{10y}$

or, $8y+16=10y$ [squaring again]

or, $2y=16$ or, $y=8$

or, $x^2-6x+13=8$ [putting the value of y]

or, $x^2-6x+5=0$ or, $(x-1)(x-5)=0$

$\therefore$ $x=1$ or, 5.

Verification: For $x=1$, left-hand side = $\sqrt{10}-\sqrt{8}=$ right-hand side.

For $x=5$, left-hand side = $\sqrt{10}-\sqrt{8}=$ right-hand side.

$\therefore$ Required solution, $x=1, 5$.

Example 6. Solve: $(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}=2^{\frac{1}{3}}$

Solution: $(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}=2^{\frac{1}{3}}$

or, $1+x+1-x+3(1+x)^{\frac{1}{3}}(1-x)^{\frac{1}{3}}\left\{(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}\right\}=2$ [cubing]

or, $2+3(1+x)^{\frac{1}{3}}(1-x)^{\frac{1}{3}}2^{\frac{1}{3}}=2$

or, $3.2^{\frac{1}{3}}(1+x)^{\frac{1}{3}}.(1-x)^{\frac{1}{3}}=0$

or, $(1+x)^{\frac{1}{3}}(1-x)^{\frac{1}{3}}=0$

or, $(1+x)(1-x)=0$ [cubing again]

$\therefore$ $x=1$ or, $x=-1$

Both the roots satisfy the equation.

$\therefore$ Required solution, $x=\pm1$.

Exercise 5.2

Solve :

1. $\sqrt{x-4}+2=\sqrt{x+12}$
2. $\sqrt{11x-6}=\sqrt{4x+5}-\sqrt{x-1}$
3. $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}$
4. $\sqrt{x+4}+\sqrt{x+11}=\sqrt{8x+9}$
5. $\sqrt{11x-6}=\sqrt{4x+5}+\sqrt{x-1}$
6. $\sqrt{x^2+4x-4}+\sqrt{x^2+4x-10}=6$
7. $\sqrt{x^2-6x+9}-\sqrt{x^2-6x+6}=1$
8. $\sqrt{2x^2+5x-2}-\sqrt{2x^2+5x-9}=1$
9. $6\sqrt{\left(\frac{2x}{x-1}\right)}+5\sqrt{\left(\frac{x-1}{2x}\right)}=13$
10. $\sqrt{\left(\frac{x-1}{3x+2}\right)}+2\sqrt{\left(\frac{3x+2}{x-1}\right)}=3$

6.2 Indicial or Exponential equations

The equation in which the unknown variable exists as an index/ exponent is called an Indicial or Exponential equation. $2^x=8, 16^x=4^{x+2}. 2^{x+1}-2^x-8=0$ etc. are indicial equations, where x is an unknown variable. To solve indicial equations, the following property of indices is often used.

If $a\neq 1$, then $a^x=a^m$ if and only if $x=m$. That is why both sides of an equation are expressed in powers of the same number.

Activity: 1. Express 4096 in powers of $\frac{1}{2}$, 2, 4, 8, 16, $2\sqrt{2}$, $\sqrt[3]{4}$.

2. Express 729 in powers of 3, 9, 27, 16, $\sqrt[5]{9}$.

3. Write $\frac{64}{729}$ in powers of $\frac{3}{2}$, $\sqrt[3]{\frac{3}{2}}$.

Example 1. Solve: $2^{x+7}=4^{x+2}$

Solution: $2^{x+7}=4^{x+2}$

or, $2^{x+7}=\left(2^2\right)^{x+2}$ or, $2^{x+7}=2^{2x+4}$

$\therefore$ $x+7=2x+4$ or, $x=3$

$\therefore$ Required solution, $x=3$.

Example 2. Solve: $3.27^x=9^{x+4}$

Solution: $3.27^x=9^{x+4}$

or, $3.\left(3^3\right)^x=\left(3^2\right)^{x+4}$ or, $3.3^{3x}=3^{2(x+4)}$

or, $3^{3x+1}=3^{2(x+4)}$ or, $3^{3x+1}=3^{2x+8}$

$\therefore$ $3x+1=2x+8$ or, $x=7$

$\therefore$ Required solution, $x=7$.

Example 3. Solve: $3^{mx-1} = 3a^{mx-2}$, $(a > 0,\ a \neq 3,\ m \neq 0)$

Solution: $3^{mx-1} = 3a^{mx-2}$

or, $\frac{3^{mx-1}}{3} = 3a^{mx-1}$ [dividing both sides by 3]

or, $3^{mx-2} = a^{mx-2}$ or, $\left(\frac{a}{3}\right)^{mx-2} = 1 = \left(\frac{a}{3}\right)^0$

$\therefore$ $mx - 2 = 0$ or, $mx = 2$ or, $x = \frac{2}{m}$

$\therefore$ Required solution, $x = \frac{2}{m}$.

Example 4. Solve: $2^{3x-5}.\ a^{x-2} = 2^{x-3}.2a^{1-x}$, $(a > 0$ and $a \neq \frac{1}{2})$

Solution: $2^{3x-5}.\ a^{x-2} = 2^{x-3}.2a^{1-x}$

or, $\frac{a^{x-2}}{a^{1-x}} = \frac{2^{x-3}.2^1}{2^{3x-5}}$ or, $a^{x-2-1+x} = 2^{x-3+1-3x+5}$

or, $a^{2x-3} = 2^{-2x+3}$ or, $a^{2x-5} = 2^{-(2x-3)}$

or, $a^{2x-3} = \frac{1}{2^{2x-3}}$ or, $a^{2x-3}.2^{2x-3} = 1$

or, $(2a)^{2x-3} = 1 = (2a)^0$ $\therefore$ $2x - 3 = 0$

or, $2x = 3$ or, $x = \frac{3}{2}$

$\therefore$ Required solution, $x = \frac{3}{2}$.

Example 5. Solve: $a^{-x}\left(a^x + b^{-x}\right) = \frac{a^2b^2 + 1}{a^2b^2}$, $(a > 0, b > 0$ and $ab \neq 1)$

Solution: $a^{-x}\left(a^x + b^{-x}\right) = 1 + \frac{1}{a^2b^2}$ or, $a^{-x}.a^x + a^{-x}.b^{-x} = 1 + \frac{1}{a^2b^2}$

or, $1 + (ab)^{-x} = 1 + (ab)^{-2}$ or, $(ab)^{-x} = (ab)^{-2}$

$\therefore$ $-x = -2$ or, $x = 2$

$\therefore$ Required solution, $x = 2$.

Example 6. Solve: $3^{x+5} = 3^{x+3} + \frac{8}{3}$

Solution: $3^{x+5} = 3^{x+3} + \frac{8}{3}$ or, $3^x.3^5 = 3^x3^3 + \frac{8}{3}$

or, $3^x.3^6 - 3^x.3^4 = 8$ [multiplying both sides by 3 and then transposing]
or, $3^x.3^4(3^2 - 1) = 8$ or, $3^{x+4}.8 = 8$
or, $3^{x+4} = 1 = 3^0$ $\therefore$ $x + 4 = 0$ or, $x = -4$
$\therefore$ Required solution, $x = -4$.

Example 7. Solve: $3^{2x-2} - 5.3^{x-2} - 66 = 0$

Solution: $3^{2x-2} - 5.3^{x-2} - 66 = 0$ or, $\frac{3^{2x}}{9} - \frac{5}{9}.3^x - 66 = 0$
or, $3^{2x} - 5.3^x - 594 = 0$ [multiplying both sides by 9]
or, $a^2 - 5a - 594 = 0$ [taking $3^x = a$]
or, $(a - 27)(a + 22) = 0$
Now, $a \neq -22$, since $a = 3^x > 0$. $\therefore$ $a + 22 \neq 0$.
Thus, $a - 27 = 0$ or, $3^x = 27 = 3^3$ $\therefore$ $x = 3$
$\therefore$ Required solution, $x = 3$.

Example 8. Solve: $a^{2x} - (a^3 + a)a^{x-1} + a^2 = 0$ $(a > 0, a \neq 1)$

Solution: $a^{2x} - (a^3 + a)a^{x-1} + a^2 = 0$
or, $a^{2x} - a(a^2 + 1)a^x.a^{-1} + a^2 = 0$ or, $a^{2x} - (a^2 + 1)a^x + a^2 = 0$
or, $p^2 - (a^2 + 1)p + a^2 = 0$ [taking $a^x = p$]
or, $p^2 - a^2p - p + a^2 = 0$ or, $(p - 1)(p - a^2) = 0$
$\therefore$ $p = 1$ or, $p = a^2$
or, $a^x = 1 = a^0$ or, $a^x = a^2$
$\therefore$ $x = 0$ $\therefore$ $x = 2$
$\therefore$ Required solution, $x = 0, 2$.

Exercise 5.3

Solve :

1. $3^{x+2} = 81$
2. $5^{3x-7} = 3^{3x-7}$
3. $2^{x-4} = 4a^{x-6}$, $(a > 0, a \neq 2)$
4. $(\sqrt{3})^{x+5} = (\sqrt[3]{3})^{2x+5}$
5. $(\sqrt[5]{4})^{4x+7} = (\sqrt[11]{64})^{2x+7}$
6. $\frac{3^{3x-4}.a^{2x-5}}{3^{x+1}} = a^{2x-5}$ $(a > 0)$
7. $\frac{5^{3x-5}.b^{2x-6}}{5^{x+1}} = a^{2x-6}(a > 0, b > 0, 5b \neq a)$
8. $4^{x+2} = 2^{2x+1} + 14$
9. $5^x + 5^{2-x} = 26$
10. $3(9^x - 4.3^{x-1}) + 1 = 0$
11. $4^{1+x} + 4^{1-x} = 10$
12. $2^{2x} - 3.2^{x+2} = -32$

5.4 System of quadratic equations with two variables

The method of solution of the system of two linear equations with two variables or the system of two equations of which one is linear and the other quadratic with two variables have been discussed in the Secondary Algebra Book. Here we will discuss the solution methodology of some systems involving two such quadratic equations. It may be mentioned that if x and y are two variables of any system, then $(x, y) = (a, b)$ is a solution of this system when both sides of the two equations will be equal if we substitute a for x and b for y.

Example 1. Solve: $x + \frac{1}{y} = \frac{3}{2}, \quad y + \frac{1}{x} = 3$

Solution: $x + \frac{1}{y} = \frac{3}{2} \quad (i), \qquad y + \frac{1}{x} = 3 \qquad (ii)$

From (i) we get, $xy + 1 = \frac{3}{2}y \qquad (iii)$ and from (ii), $xy + 1 = 3x \qquad (iv)$

(iii) and (iv) give $\frac{3}{2}y = 3x$ or, $y = 2x \quad (v)$

Putting the value of y from (v) in (iv), we obtain

$2x^2 + 1 = 3x$ or, $2x^2 - 3x + 1 = 0$

or, $(x-1)(2x-1) = 0 \quad \therefore \ x = 1$ or $\frac{1}{2}$

When $x = 1, \ \frac{1}{2}$ we get from (v) $y = 2$ and 1 respectively.

$\therefore$ Required solution: $(x, y) = (1, 2), \left(\frac{1}{2}, 1\right)$.

Example 2. Solve: $x^2 = 3x + 6y, \ xy = 5x + 4y$

Solution: $x^2 = 3x + 6y \qquad (i) \qquad xy = 5x + 4y \qquad (ii)$

Subtracting (ii) from (i), $x(x-y) = -2(x-y)$

or, $x(x-y) + 2(x-y) = 0$ or, $(x-y)(x+2) = 0$

or, $\therefore x = y \qquad (iii), \qquad x = -2 \qquad (iv)$

We get from (iii) and (i), $y^2 = 9y$ or, $y(y-9) = 0 \ \therefore y = 0$ or, 9

From (iii) we have, $x = 0$ when $y = 0$ and $x = 9$ when $y = 9$.

Again from (iv) and (i) we get, $4 = -6 + 6y$ or, $6y = 10$ or, $y = \frac{5}{3}$

$\therefore$ Required solution: $(x, y) = (0, 0), (9, 9), \left(-2, \frac{5}{3}\right)$.

Example 3. Solve: $x^2 + y^2 = 61,\ xy = -30$

Solution: $x^2 + y^2 = 61 \qquad (i) \qquad xy = -30 \qquad (ii)$

Multiplying (ii) by 2 and subtracting from (i) we get,

$(x-y)^2 = 121$ or, $x - y = \pm 11 \qquad (iii)$

Again multiplying (ii) by 2 and adding with (i) we have,

$(x+y)^2 = 1$ or, $x + y = \pm 1 \qquad (iv)$

Now from (iii) and (iv), we can write

$$\left.\begin{array}{r} x+y=1 \\ x-y=11 \end{array}\right\} (v), \quad \left.\begin{array}{r} x+y=1 \\ x-y=-11 \end{array}\right\} (vi), \quad \left.\begin{array}{r} x+y=-1 \\ x-y=11 \end{array}\right\} (vii), \quad \left.\begin{array}{r} x+y=-1 \\ x-y=-11 \end{array}\right\} (viii)$$

Solving we get,

From (v), $x = 6,\ y = -5$; from (vi), $x = -5,\ y = 6$

from (vii), $x = 5,\ y = -6$ from (viii), $x = -6,\ y = 5$

$\therefore$ Required solution $(x,\ y) = (6,\ -5), (-5,\ 6), (5,\ -6), (-6,\ 5)$.

Example 4. Solve: $x^2 - 2xy + 8y^2 = 8,\ 3xy - 2y^2 = 4$

Solution: $x^2 - 2xy + 8y^2 = 8, \qquad (i) \qquad 3xy - 2y^2 = 4 \qquad (ii)$

Dividing (i) by (ii) we get,

$\dfrac{x^2 - 2xy + 8y^2}{3xy - 2y^2} = \dfrac{2}{1}$ or, $x^2 - 2xy + 8y^2 = 6xy - 4y^2$

or, $x^2 - 8xy + 12y^2 = 0$ or, $x^2 - 6xy + 2xy + 12y^2 = 0$

or, $(x - 6y)(x - 2y) = 0 \quad \therefore x = 6y \qquad (iii)$ or, $x = 2y \qquad (iv)$

Putting the value of x from (iii) in (ii), we obtain

$3.6y.y - 2y^2 = 4$ or, $16y^2 = 4$ or, $y^2 = \dfrac{1}{4}$ or, $y = \pm\dfrac{1}{2}$

From (iii), $x = 6 \times \left(\pm\dfrac{1}{2}\right) = \pm 3.$

Again putting the value of x from (iv) in (ii), we obtain

$3.2y.y - 2y^2 = 4$ or, $4y^2 = 4$ or, $y^2 = 1$ or, $y = \pm 1$

From (iv) we get, $x = 2 \times (\pm 1) = \pm 2$.

$\therefore$ Required solution: $(x, y) = \left(3,\ \dfrac{1}{2}\right), \left(-3,\ -\dfrac{1}{2}\right), (2,\ 1), (-2,\ -1).$

Example 5. Solve: $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{5}{2}, \quad x^2 + y^2 = 90$

Solution: $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{5}{2}$ $\quad(i)\quad x^2+y^2=90\quad(ii)$

From (i) we get,

$\dfrac{(x+y)^2+(x-y)^2}{(x+y)(x-y)}=\dfrac{5}{2}$ or, $\dfrac{2(x^2+y^2)}{x^2-y^2}=\dfrac{5}{2}$

or, $\dfrac{2\times 90}{x^2-y^2}=\dfrac{5}{2}$ [putting $x^2+y^2=90$ from (ii)]

$\therefore\ x^2-y^2=72 \quad (iii)$

(ii) + (iii) implies, $2x^2=162$ or, $x^2=81$ or, $x=\pm 9$

and $(ii)-(iii)$ implies, $2y^2=18$ or, $y^2=9$ or, $y=\pm 3$

$\therefore$ Required solution: $(x, y)=(9,3), (9, -3), (-9, 3), (-9, -3)$.

Exercise 5.4

Solve :

1. $(2x+3)(y-1)=14,\ (x-3)(y-2)=1$
2. $(x-2)(y-1)=3,\ (x+2)(2y-5)=15$
3. $x^2=7x+6y,\ y^2=7y+6x$
4. $x^2=73x+2y,\ y^2=3y+2x$
5. $x+\dfrac{4}{y}=1,\ y+\dfrac{4}{x}=25$
6. $y+3=\dfrac{4}{x},\ x-4=\dfrac{5}{3y}$
7. $xy-x^2=1,\ y^2-xy=2$
8. $x^2-xy=14,\ y^2+xy=60$
9. $x^2+y^2=25,\ xy=12$
10. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{10}{3},\ x^2-y^2=3$
11. $x^2+xy+y^2=3,\ x^2-xy+y^2=7$
12. $2x^2+3xy+y^2=20,\ 5x^2+4y^2=41$

5.5 Applications of simultaneous quadratic equations

Many problems of everyday life can be solved using the knowledge of simultaneous equations. Sometimes, we are to find out the values of two unknown quantities of the problem. In that case, the two unknown quantities are tken as *x* and *y* or any other symbols. Then consistent and independent equations are formed according to the conditions of the problem. The values of the unknown quantities *x* and *y* are obtained by solving these equations.

Example 1. The sum of the areas of the two square regions is 650 square metres. If the area of the rectangular region formed by the two sides of the two squares is 323 square metres, what are the lengths of the sides of the two squares?

Solution: Suppose, the length of the side of one square is x metres and that of the other square is y metres.

According to the question, $x^2+y^2=650$ ....(i)

and $xy=323$ ....(ii)

$\therefore (x+y)^2 = x^2+y^2+2xy=650+646=1296$

i.e., $(x+y)=\pm\sqrt{1296}=\pm 36$

and $(x-y)^2 = x^2+y^2-2xy=650-646=4$

i.e., $(x-y)=\pm 2$

Since the length is positive, the value of $(x+y)$ must be positive.

$\therefore (x+y)=36$......................(iii)

$(x-y)=\pm 2$.............................(iv)

Adding (iii) and (iv), $2x=36\pm 2$

$\therefore x=\dfrac{36\pm 2}{2}=18\pm 1=19$ or, 17

From equation (iii) , $y=36-x=17$ or, 19.

$\therefore$ The length of the side of one square is 19 metres and that of the other square is 17 metres.

Example 2. Twice the breadth of a rectangle is 10 metres more than its length. If the area of the region enclosed by the rectangle is 600 square metres, find its length.

Solution: Suppose that the length of the rectangle $= x$ metres and the breadth of the rectangle $= y$ metres.

According to the question, $2y = x+10$ ....(i)

$xy=600$......................(ii)

From equation (i) , $y=\dfrac{10+x}{2}$

Putting the value of y in equation (ii) we get, $\dfrac{x(10+x)}{2}=600$

or, $\dfrac{10x+x^2}{2}=600$ or, $x^2+10x=1200$

or, $x^2+10x-1200=0$ or, $(x+40)(x-30)=0$

$\therefore$ $(x+40)=0$ or, $(x-30)=0$

That is, $x=-40$ or, $x=30$

But length cannot be negative,

$\therefore x=30$

Hence, the length of the rectangle = 30 metres.

Example 3. If a number of two digits be divided by the product of its digits, the quotient is 3. When 18 is added to the number, the digits of the number change their places. Find the number.

Solution: Suppose, tens places digit $= x$ meters and ones places digit $= y$ metres.

$\therefore$ The number = $10x + y$

From the 1st condition, $\frac{10x+y}{xy} = 3$ or, $10x + y = 3xy$ ....(i)

From the 2nd condition, $10x + y + 18 = 10y + x$ or, $9x - 9y + 18 = 0$

or, $x - y + 2 = 0$ or, $y = x + 2$........................(ii)

Putting $y = x + 2$ in equation (i) we get $10x + x + 2 = 3.x(x+2)$

or, $11x + 2 = 3.x^2 + 6x$ or, $3x^2 - 5x - 2 = 0$

or, $3x^2 - 6x + x - 2 = 0$ or, $3x(x-2) + 1(x-2) = 0$

or, $(x-2)(3x+1) = 0$

Therefore, $(x-2) = 0$ or, $(3x+1) = 0$ or, $3x = -1$

$\therefore$ $x = 2$ or, $x = -\frac{1}{3}$

But the degit or a number cannot be negative or fraction.

Therefore, $x = 2$ and $y = x + 2 = 2 + 2 = 4$

$\therefore$ The required number = 24 .

Exercise 5.5

1. The sum of the areas of two square regions is 481 square metres; if the area of the rectangle formed by the two sides of the two squares is 240 square metres, what is the length of a side of each of the squares?
2. The sum of squares of two positive numbers is 250; the product of the numbers is 117; find the two numbers.
3. The length of a diagonal of a rectangle is 10 metres. The area of a rectangle whose sides are the sum and difference of the sides of the former one is 28 square metres. Find the length and breadth of the former rectangle.
4. The sum of squares of two numbers is 181 and the product of the numbers is 90. Find the difference of the squares of the two numbers.
5. The area enclosed by a rectangle is 24 square metres. The length and breadth of another rectangle are respectively 4 metres and 1 metre more than the length and breadth of the first rectangle and the area enclosed by the later rectangle is 50 square metres. Find the length and bredth of the first rectangle.

6. Twice the breadth of a rectangle is 23 metres more than its length. If the area enclosed by the rectangle is 600 square metres, find the length and breadth of the rectangle.
7. The perimeter of a rectangle is 8 metres more than the sum of its diagonals. If the area enclosed by the rectangle is 48 square metres; find its length and breadth.
8. If a number of two digits be divided by the product of its digits, the quotient is 2. When 27 is added to the number, the digits in the number change their places. Find the number.
9. The perimeter of a rectangular garden is 56 metres and one diagonal is 20 metres. What is the length of the side of the square which encloses an area equal to the area of that garden?
10. The area of a rectangular field is 300 square metres and its semiperimeter is 10 metres more than a diagonal. Find the length and bredth of the rectangular field.

5.6 System of identical equations with two variables

The method of solution of identical equations with one variable has been discussed in the previous chapter. Here we will discuss the solution methodology of the system of indicial equations with two variables.

Example 1. Solve: $a^{x+2}.\, a^{2y+1} = a^{10}, \quad a^{2x}.a^{y+1} = a^9 \, (a \neq 1)$

Solution: $a^{x+2}.\, a^{2y+1} = a^{10} \quad (i) \qquad a^{2x}.a^{y+1} = a^9 \quad (ii)$

From (i), $a^{x+2y+3} = a^{10}$ or, $x + 2y + 3 = 10$ or, $x + 2y - 7 = 0$ (iii)

From (ii), $a^{2x+y+1} = a^9$ or, $2x + y + 1 = 9$ or, $2x + y - 8 = 0$ (iv)

From (iii) and (iv), by cross multiplication,

$$\frac{x}{-16+7} = \frac{y}{-14+8} = \frac{1}{1-4}$$

or, $\dfrac{x}{-9} = \dfrac{y}{-6} = \dfrac{1}{-3}$

or, $\dfrac{x}{3} = \dfrac{y}{2} = 1$

or, $x = 3,\ y = 2$

$\therefore$ Required solution: $(x, y) = (3, 2)$.

Example 2. Solve: $3^{3y-1} = 9^{x+y}, \quad 4^{x+3y} = 16^{2x+3}$

Solution: $3^{3y-1} = 9^{x+y} \quad (i)$

or, $3^{3y-1} = (3^2)^{x+y} = 3^{2x+2y}$

$\therefore\ 3y-1=2x+2y$ or, $2x-y+1=0$ (iii) $4^{x+3y}=16^{2x+3}$ (ii)

or, $4^{x+3y}=\left(4^2\right)^{2x+3}$ or, $4^{x+3y}=4^{4x+6}$ or, $x+3y=4x+6$

or, $3x-3y+6=0$ or, $x-y+2=0$ (iv)

From (iii) and (iv), by cross multiplication,

$$\frac{x}{-2+1}=\frac{y}{1-4}=\frac{1}{-2+1}$$

or, $\dfrac{x}{-1}=\dfrac{y}{-3}=-1$

or, $x=1,\ y=3$

$\therefore$ Required solution: $(x,y)=(1,3)$.

Example 3. Solve: $x^y=y^x,\ x=2y$

Solution: $x^y=y^x$ (i) $x=2y$ (ii) $(x\neq 0,\ y\neq 0)$

Putting the value of x from (ii) in (i), we obtain $(2y)^y=y^{2y}$ or, $2^y.y^y=y^{2y}$

or, $\dfrac{y^{2y}}{y^y}=2^y$ or, $y^y=2^y$ $\therefore\ y=2$ and $x=4$ [from(ii)]

$\therefore$ Required solution: $(x,y)=(4,2)$.

Example 4. Solve: $x^y=y^2,\ y^{2y}=x^4$, when $x\neq 1$

Solution: $x^y=y^2$ (i), $y^{2y}=x^4$ (ii)

From (i), $\left(x^y\right)^y=\left(y^2\right)^y$ or, $x^{y^2}=y^{2y}$ (iii)

From (iii) and (ii), $x^{y^2}=x^4$ $\therefore\ y^2=4$ or, $y=\pm 2$

When $y=2$ then from (i) we get, $x^2=2^2=4$ or, $x=\pm 2$

Again when $y=-2$ then from (i) we get, $(x)^{-2}=(-2)^2=4$

or, $\dfrac{1}{x^2}=4$ or, $x^2=\dfrac{1}{4}$ or, $x=\pm\dfrac{1}{2}$

$\therefore$ Required solution: $(x,y)=(2,\ 2),(-2,\ 2),\left(\dfrac{1}{2},-2\right),\left(-\dfrac{1}{2},-2\right)$.

Example 5. Solve: $8.2^{xy}=4^y$, $9^x.3^{xy}=\dfrac{1}{27}$

Solution: $8.2^{xy}=4^y$ (i), $9^x.3^{xy}=\dfrac{1}{27}$ (ii)

From (i), $2^3.2^{xy}=\left(2^2\right)^y$ or, $2^{3+xy}=2^{2y}$ $\therefore\ 3+xy=2y$ (iii)

From (*ii*), $\left(3^2\right)^x.3^{xy}=\frac{1}{3^3}$ or, $3^{2x+xy}=3^{-3}$ $\therefore$ $2x+xy=-3$ (iv)

Subtracting (*iv*) from (*iii*), $3-2x=2y+3$ or, $-x=y$ (v)

Putting the value of y from (*v*) in (*iii*), we obtain, $3-x^2=-2x$

or, $x^2-2x-3=0$ or, $(x+1)(x-3)=0$ $\therefore$ $x=-1$ or, $x=3$

From (*v*), we get

for $x=-1$, $y=1$ and for $x=3$, $y=-3$.

$\therefore$ Required solution: $(x,y)=(-1,1),(3,-3)$.

Example 6. Solve: $18y^x-y^{2x}=81, 3^x=y^2$

Solution: $18y^x-y^{2x}=81,$ (i) $3^x=y^2$ (ii)

From (*i*) we get, $y^{2x}-18y^x+81=0$ or, $\left(y^x-9\right)^2=0$

or, $y^x-9=0$ or, $y^x=3^2$ (iii)

From (*ii*), $\left(3^x\right)^x=\left(y^2\right)^x$ or, $3^{x^2}=y^{2x}$ (iv)

From (*iii*) we get, $\left(y^x\right)^2=\left(3^2\right)^2$ or, $y^{2x}=3^4$ (v)

(*iv*) and (*v*) imply, $3^{x^2}=3^4$ *i.e.*, $x^2=4$ or, $x=\pm 2$

When $x=2$, we get from (*ii*), $y^2=9$ or, $y=\pm 3$

When $x=-2$, we get from (*iii*), $y^{-2}=9$ or, $y^2=\frac{1}{9}$ or, $y=\pm\frac{1}{3}$

$\therefore$ Required solution: $(x,y)=(2,3), (2,-3), \left(-2,\frac{1}{3}\right), \left(-2,-\frac{1}{3}\right)$.

Exercises 5.6

Solve :

1. $2^x+3^y=31$, $2^x-3^y=-23$
2. $3^x=9^y$, $5^{x+y+1}=25^{xy}$
3. $3^x.9^y=81$, $2x-y=8$
4. $2^x.3^y=18$, $2^{2x}.3^y=36$
5. $a^x.a^{y+1}=a^7$, $a^{2y}.a^{3x+5}=a^{20}$
6. $\left.\begin{array}{l} y^x=x^2 \\ x^{2x}=y^4 \end{array}\right\} y\neq 1$
7. $y^x=4$, $y^2=2^x$
8. $4^x=2^y$, $(27)^{xy}=9^{y+1}$
9. $8y^x-y^{2x}=16$, $2^x=y^2$

5.7 Solution of the quadratic equation $ax^2+bx+c=0$ using graph

We have already solved the quadratic equation $ax^2 + bx + c = 0$ algebraically. Method of solving it using graph will be discussed now.

Suppose $y = ax^2 + bx + c$. Then the values of x for which $y = 0$ (i.e. the graph of y intersects the x-axis) are the solutions of $ax^2 + bx + c = 0$.

Example 1. Use graph to solve $x^2 - 5x + 4 = 0$.

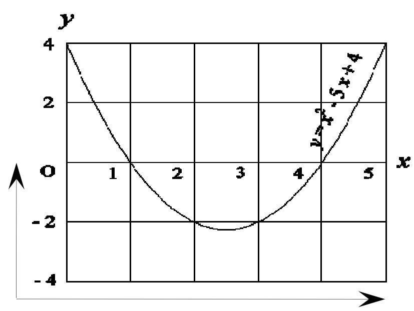
Solution: Suppose $y = x^2 - 5x + 4$.

For some values of x, we find the corresponding values of y to get the associated points on the graph and put these in the table below.

x	0	1	2	2.5	3	4	5
y	4	0	–2	–2.25	–2	0	4

We draw the graph of the equation plotting the points given in the above table. It is seen that the graph intersects the x-axis at (1, 0) and (4, 0).

Therefore, the solution of the equation is $x = 1$ or $x = 4$.



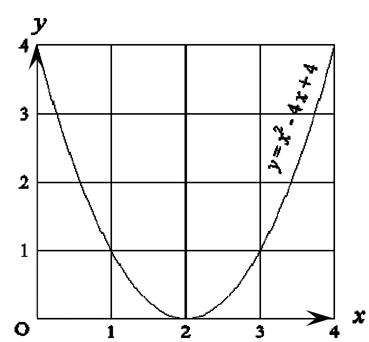
Example 2. Solve $x^2 - 4x + 4 = 0$ graphically.

Solution: Let $y = x^2 - 4x + 4$.

We find the values of y corresponding to some values of x which give the associated points for the graph:

x	0	1	1.5	2	2.5	3	4
y	4	1	0.25	0	0.25	1	4

After plotting the points given in the above table we draw the graph of the equation. We can see from the figure that the graph touches the x-axis at (2, 0). Since a quadratic equation has two roots, the solutions of the equation are $x = 2$, $x = 2$.



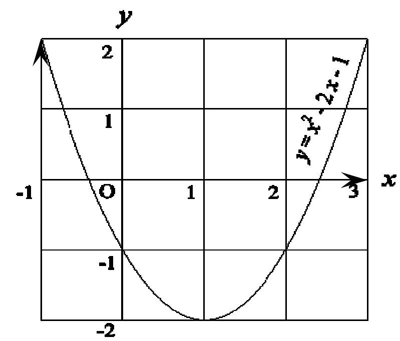
Example 3. Solve $x^2 - 2x - 1 = 0$ with the help of graph.

Solution: Let us take $y = x^2 - 2x - 1$.

For some values of *x*, we find the corresponding values of *y* that give the associated points on the graph:

x	–1	–0.5	0	0.5	1	1.5	2	2.5	3
y	2	0.25	–1	–1.75	–2	–1.75	–1	0.25	2

We sketch the graph of the equation plotting the tabulated points in the graph paper. It is observed that the graph intersects the *x*-axis approximately at (– 0.4, 0) and (2.4, 0). Therefore, the solution of the equation is $x = -0.4$ (approx.) or $x = 2.4$ (approx.).



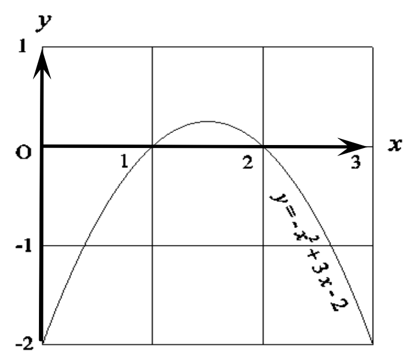
Example 4. Find both the roots of $-x^2 + 3x - 2 = 0$ with the help of graph.

Solution: Let us assume that $y = -x^2 + 3x - 2$.

For some values of *x*, we find the associated values of *y* to get the related points of the graph of the equation and put these in the table given below.

x	0	0.5	1	1.5	2	2.5	3
y	–2	–0.75	0	0.25	0	–0.75	–2

Plotting the points obtained in the graph paper, we get the graph of the equation. We can see that the graph passes through the points (1, 0) and (2, 0) on the *x*-axis. Therefore, the solution of the equation is $x = 1$ or $x = 2$.



Exercise 5.7

1. $ax^2 + bx + c = 0$ where a, b, c are real numbers. What is the value of b in the equation $x^2 - x - 12 = 0$?
   A. 0 B. 1
   C. -1 D. 3
2. What is the solution of the equation $16^x = 4^{x+1}$?
   A. 2 B. 0
   C. 4 D. 3
3. Which one is a solution of the equation $x^2 - x + 13 = 0$?
   A. $\frac{-1+\sqrt{-51}}{2}$ B. $\frac{-1-\sqrt{51}}{2}$
   C. $\frac{1+\sqrt{-51}}{2}$ D. $\frac{1+\sqrt{51}}{2}$
4. What is the correct solution of $y^x = 9$, $y^2 = 3^x$?
   A. $(2, 3), (-2, \frac{1}{9})$ B. $(2, -2), (3, \frac{1}{9})$
   C. $(2, \frac{1}{9}), (-2, 3)$ D. $(-2, -\frac{1}{9}), (2, 3)$

Answer to the question nos. 5 and 6 on the basis of the information given below:
The difference of the squares of two positive whole numbers is 11 and the product of the numbers is 30.

5. What are the numbers ?
   A. 1 and 30 B. 2 and 15
   C. 5 and 6 D. 5 and – 6
6. What is the sum of the squares of the numbers ?
   A. 1 B. 5
   C. 41 D. $\sqrt{41}$
7. The sum of a number and its multiplicative inverse is 6. The formation of equation is
   i. $x + \frac{1}{x} = 6$
   ii. $x^2 + 1 = 6x$
   iii. $x^2 - 6x - 1 = 0$
   Which one is true?
   A. i & ii B. i & iii
   C. ii & iii D. i, ii & iii

8. Which one is the solution of $2^{px-1} = 2q^{px-2}$?

A. $\frac{p}{2}$ B. p

C. $-\frac{p}{2}$ D. $\frac{2}{p}$

Solve the following equations using graph:

9. $x^2 - 4x + 3 = 0$
10. $x^2 + 2x - 3 = 0$
11. $x^2 + 7x = 0$
12. $2x^2 - 7x + 3 = 0$
13. $2x^2 - 5x + 2 = 0$
14. $x^2 + 8x + 16 = 0$
15. $x^2 + x - 3 = 0$
16. $x^2 = 8$
17. Twice the square of a number is less by 3 than 5 times of the number. But 3 times of the square of that number is greater by 3 than 5 times of the number.
    a. Form the equation using the informations given by stimulus.
    b. Solve the first equation using formula.
    c. Solve the second equation using graph.
18. The area of a land of Mr. Ashfaque Ali is 0.12 hector. One-half of its perimeter is greater by 20 metres than one of its diagonal. He sells one-third of his land to Mr. Shyam. The length of Shyam's land is greater by 5 metres than its breadth.
    a. Form two equations in the light of stimulus.
    b. Find the length and breadth of the land of Mr. Ashfaque Ali.
    c. Find the length of a diagonal and the perimeter of the land of Mr. Shyam.

Chapter six

Inequality

We have acquired knowledge about equation or equality. But inequality also has an important and extensive role in practical life. Among all the things which are seen everyday in nature, measure of any kind of any two like objects, animals or men are not the same; even these objects, animals or men are not like looking. Due to this reason, we need the knowledge of inequality.

At the end of this chapter, students will be able to

- explain the inequality of one or two variables
- form and solve simple inequalities of two variables
- use inequalities to solve practical mathematical problems
- solve inequalities of one and two variables graphically.

Inequality

Suppose, there are 200 students in a class. Obviously, all the students do not come to the class everyday. If the number of present student is x in a particular day we can write $0 < x \leq 200$. In a similar situation, we see that not all the invited persons become present in a ceremony. Clear conception of inequalities is needed in making dresses and other consumer goods. Essential materials for constructing buildings, printing books and for many other similar works cannot be estimated exactly. So, initially we have to buy or collect those essential materials on the basis of guess. Therefore, it is understood that the knowledge of inequalities is very essential in our everyday life.

Postulates or rules related to equations are also applicable to inequality. With the exception that if the unequal quantities are multiplied or divided by equal negative numbers then the direction of inequality is reversed.

Let us consider the inequality $4 < 6$. We have

$4 + 2 < 6 + 2$ or, $6 < 8$ [adding 2 to both sides]

We also get,

$2 < 4$ [subtracting 2 from both sides]

$4 < 12$ [multiplying both sides by 2]

$2 < 3$ [dividing both sides by 2]

Multiplying separately both sides of the inequality by -2, we get -8 and -12.

Here, $-8 > -12$.

Similarly, $-2 > -3$ [dividing both sides by -2]

Generally it may be stated that if $a < b$, then

$a + c < b + c$ for all values of c

	$a-c<b-c$	for all values of c
	$ac<bc$	for positive values of c
	$\frac{a}{c}<\frac{b}{c}$	for positive values of c
But	$ac>bc$	for negative values of c
	$\frac{a}{c}>\frac{b}{c}$	for negative values of c.

> **Activity:** 1. Express the heights of the students in your class which are less and greater than 5 feet in terms of an inequality.
> 2. In an examination the total marks is 1000. Express the marks obtained by an examinee in the form of an inequality.

Example 1. Solve and show the solution set on a number line: $4x+4>16$.

Solution: Given that, $4x+4>16$

$\therefore 4x+4-4>16-4$ [Subtracting 4 from both sides]

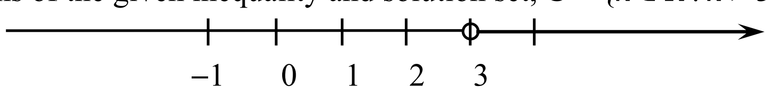
or, $4x>12$ or, $\frac{4x}{4}>\frac{12}{4}$ [dividing both sides by 3]

or, $x>3$

$\therefore$ The required solution: $x>3$.

Here the solution set $S=\{x\in R: x>3\}$.

The solution set is shown on the number line below. All real numbers greater than 4 are solutions of the given inequality and solution set, $S=\{x\in R: x>3\}$.



Example 2. Solve and show the solution set on a number line: $x-9>3x+1$.

Solution : Given that, $x-9>3x+1$

$\therefore x-9+9>3x+1+9$ or, $x>3x+10$

or, $x-3x>3x+10-3x$ or, $-2x>10$

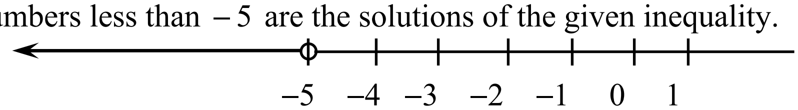
or, $\frac{-2x}{-2}<\frac{10}{-2}$ [the direction of inequality is reversed on division of both sides by negative integer –2]

or, $x<-5$

$\therefore$ The required solution: $x<-5$.

Here the solution set $S=\{x\in R: x<-5\}$.

All real numbers less than -5 are the solutions of the given inequality.



N.B. The solution of an inequality are usually expressed by an inequality like solution of an equation is expressed by an equation (equality). The solution set of an inequality is usually an infinite subset of the set of real numbers. $a \geq b$ means $a > b$ or $a = b$. i.e. $a \geq b$ is false only if $a < b$.
Therefore, both the statements $4 > 3$ and $4 \geq 4$ are correct.
Example 3. Solve: $a(x+b) < c$, $[a \neq 0]$

Solution: If a is positive, then $\frac{a(x+b)}{a} < \frac{c}{a}$ [dividing both sides by a]

or, $x + b < \frac{c}{a}$ or, $x < \frac{c}{a} - b$

If a is negative, then by the same process we get, $\frac{a(x+b)}{a} > \frac{c}{a}$

or, $x + b > \frac{c}{a}$ or, $x > \frac{c}{a} - b$

$\therefore$ Required solution: $(i)\ x < \frac{c}{a} - b$, if $a > 0$,

$(ii)\ x > \frac{c}{a} - b$, if $a < 0$.

N.B. If $a = 0$ and c is positive, then the inequality holds for any value of x. But if $a = 0$ and c is negative or zero, then the inequality has no solution.

Exercise 6.1

Solve the inequalities and show the solution set on a number line:

1. $y - 3 < 5$
2. $3(x-2) < 6$
3. $3x - 2 > 2x - 1$
4. $z \leq \frac{1}{2}z + 3$
5. $8 \geq 2 - 2x$
6. $x \leq \frac{x}{3} + 4$
7. $5(3-2t) \leq 3(4-3t)$
8. $\frac{x}{3} + \frac{x}{4} + \frac{x}{5} > \frac{47}{60}$

Application of inequalities:

You have learnt to solve problems using equations. Following the same procedure, you will be able to solve the problems involving some inequality.

Example 1. In an examination, Rama obtained $5x$ and $6x$ marks and Kumkum obtained $4x$ and 84 marks in Bangla first and second paper respectively. None of them secured less than 40 marks in any paper. Kumkum secured the first position and Rama secured the second position in Bangla. Express the possible values of x using inequalities.

Solution: The total marks obtained by Rama and Kumkum in Bangla are $5x + 6x$ and $4x + 84$ respectively.

According to the problem, $5x+6x<4x+84$ or, $5x+6x<4x+84$

or, $7x<84$ or, $x<\frac{84}{7}$

or, $x<12$

But, $4x\geq 40$ [obtained minimum marks are 40] or, $x\geq 10$

$\therefore$ In terms of inequalities we can write $10\leq x\leq 12$.

Example 2. A student has bought x pencils at Tk. 5 each and $(x+4)$ khatas at Tk. 8 each. If the total cost does not exceed Tk. 97, what is the maximum number of pencils he has bought?

Solution : The price of x pencils is Tk. $5x$ and that of $(x+4)$ khatas is Tk. $8(x+4)$.

According to the problem,

$5x+8(x+4)\leq 97$ or, $5x+8x+32\leq 97$

or, $13x\leq 97-32$ or, $13x\leq 65$ or, $x\leq \frac{65}{13}$

or, $x\leq 5$

$\therefore$ The maximum number of pencils the student has bought is 5.

Activity : David purchase x kg apples by the rate of 140 Tk. He gives the seller a note of 1000 Tk. The seller returns him x number notes of 50 Tk. Express the problem in inequalities and find out the probable value of x.

Exercise 6.2

Express the problems 1- 5 in terms of inequalities and find the possible values of x :

1. A boy walked 3 hours at the rate of x km/hour and ran $\frac{1}{2}$ hour at the rate of $(x+2)$ km/hour, and the distance covered by him was less than 29 km.
2. A boarding house requires $4x$ kg of rice and $(x-3)$ kg of pulses every day and it does not require more than 40 kg of rice and pulses in total.
3. Mr. Sohrab bought x kg mango at the rate of Tk. 70 per kg. He gave two notes Tk. 500 to the seller. The seller returned the balance which included x notes of Tk. 20.
4. A car runs x km. in 4 hours and $(x+120)$ km in 5 hours. The average speed of the car does not exceed 100 km/hour.
5. The area of a piece of paper is 40 sq cm. A rectangular piece of x cm. long and 5 cm wide is cut off from it.

6. The age of the son is one-third of that of the mother. The father is 6 years older than the mother. The sum of the ages of these three persons is not more than 90 years. Express the age of the father in terms of an inequality.
7. Jeny appeared at the junior scholarship examination at the age of 14 years. She will appear at the S.S.C. examination at the age of 17 years. Express her present age in terms of an inequality.
8. The speed of a jet-plane does not exceed 300 meters/sec. Express the time required by the plane to cover 15 km in the form of an inequality.
9. The air distance of jedda from Dhaka is 5000 km. The maximum speed of a jet-plane is 900 km/hour. But on way from Dhaka to jedda, it faces air flowing at 100 km/hour from opposite direction. Express the time required for the non-stop flight from Dhaka to jedda in terms of an inequality.
10. On the basis of the above problem, express the time required for the non-stop flight from Jedda to Dhaka in the form of an inequality.
11. 5 times a positive integer is less than the sum of twice the number and 15. Express the possible values of the number in the form of an inequality.

Linear inequality with two variables:

We have learned to draw the graph of the linear equations with two variables of the form y = mx + c (whose general form is ax + by + c = 0) (see Algebra Books of class seven and eight). We have seen that the graph of each equation of this type is a straight line.

In *xy*-plane, co-ordinates of any point on the graph of equation ax + by + c = 0 satisfies the equation, that is, the left hand side of the equation will be zero if we replace the *x* and *y* by the abscissa and ordinate respectively of that point.

On the other hand, the co-ordinates of any point outside the graph does not satisfy the equation, in other words for abscissa and ordinate of that point the value of ax + by + c = 0 is greater or less than zero. When x and y of the expression ax + by + c are replaced respectively by the abscissa and ordinate of any point P on the plane is called the value of the expression at the point P and generally, that value is denoted by f(P). If P is on the graph, then f(P) = 0 and if P lies outside the graph, then either f(P) > 0 or, f(P) < 0. As a matter of fact, all points outside the graph is divided into two half-planes by the graph.

For each point P of one half plane, f(P) > 0 and for each point P of the other half plane, f(P) < 0.

In fact, f (P) = 0 for each point P on the graph.

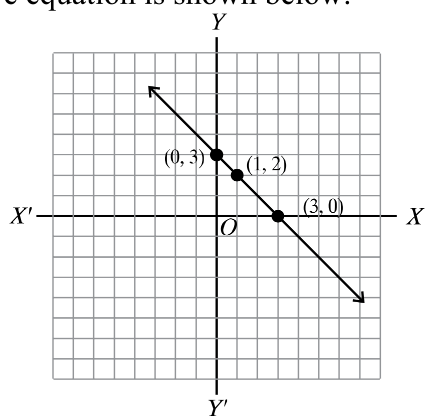
Example 1. We consider the equation $x + y - 3 = 0$.

From the equation we get, $y = 3 - x$.

Here,

x	0	3	1
y	3	0	2

On the *xy*-plane, taking the length of the side of a small square of the graph paper as unit the graph of the above equation is shown below:



This graph-line divides the plane into three parts. These are:

(1) Points on the side marked (a) of the line
(2) Points on the side marked (b) of the line
(3) Points which lie on the line.

Here, the side marked (a) may be called the upper part of the graph-line and the side marked (b) may be called the lower part of the graph-line.

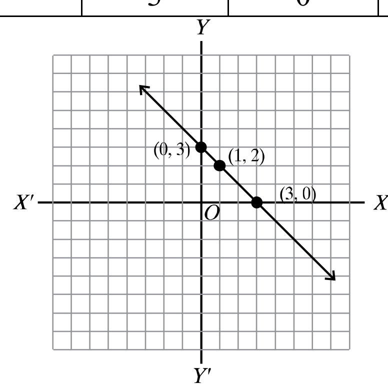
Graph of Inequalities with two variables

Example 2. Draw the graph of the inequality $x + y - 3 > 0$ or, $x + y - 3 < 0$.

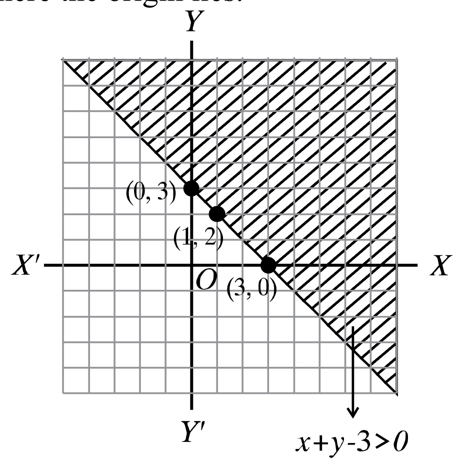
Solution: To draw the graph of the above inequalities, we first draw the graph of the equation $x + y - 3 = 0$ on the squared paper.

From the equation $x + y - 3 = 0$, we get

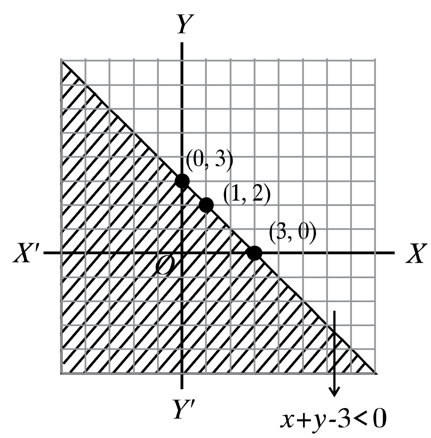
x	0	3	1
y	3	0	2



To draw the graph of the inequality $x+y-3>0$, we put the values of (x, y) at the origin (0, 0) in the inequality and get $-3>0$ which is not true. So the graph of the inequality $x+y-3>0$ will be on the side of the equation $x+y-3=0$ which is opposite to the side where the origin lies.



To draw the graph of the inequality $x+y-3<0$, we put the values of (x, y) at the origin (0, 0) in the inequality and get $-3<0$ which is true. So the graph of the inequality $x+y-3<0$ will be on the same side of the equation $x+y-3=0$ where the origin lies.



Example 3. Describe the solution set and draw the graph of the inequality $2x-3y+6\geq 0$

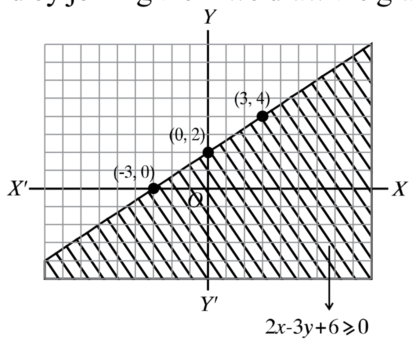
Solution: First draw the graph of the equation $2x-3y+6=0$

From the above equation we have $3y=2x-6$ or, $y=\frac{2x}{3}+2$

Co-ordinates of some points on this graph-line are

x	0	–3	3
y	2	0	4

Now on the squared paper taking length of the side of a small square as unit we plot the points (0, 2), (–3, 0), (3, 4) and by joining them we draw the graph of the equation.



Now at the origin (0, 0), the value of the expression $2x - 3y + 6$ is 6, which is positive.

Thus we have $2x - 3y + 6 > 0$ for all points on the origin side of the graph-line.

Therefore, the solution set of the inequality $2x - 3y + 6 \geq 0$ consists of the co-ordinates of all points on the graph line of the equation $2x - 3y + 6 = 0$ and the co-ordinates of all points on the origin side of the graph line.

The graph of this solution set is the shaded region of the above figure in which the graph-line is also included.

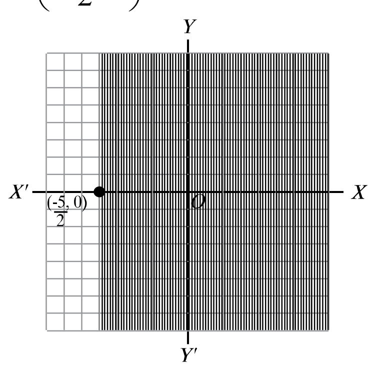
Example 4. In *xy*-plane, draw the graph of the inequality $-2x < 5$.

Solution : The inequality $-2x < 5$ can be written as

$$2x + 5 > 0 \quad \text{or, } 2x > -5 \quad \text{or, } x > -\frac{5}{2}$$

Now in *xy*-plane we draw the graph of the equation $x = -\frac{5}{2}$.

In squared paper, taking the length of two small squares as unit we draw the graph-line passing through the point $\left(-\frac{5}{2}, 0\right)$ and parallel to the y-axis.



The origin lies at the right side of the graph-line and at origin $x = 0$ which reduces the inequality to $0 > -\frac{5}{2}$ which is true.

Therefore, the co-ordinates of all points on the right side of the graph-line are the solution of the given inequality.

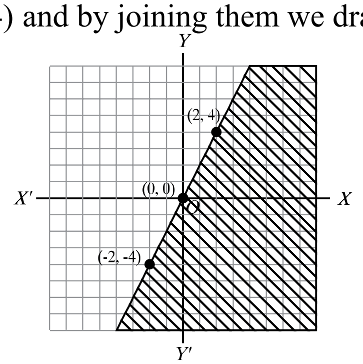
The graph of the inequality is the shaded region of the above figure (the graph of the line $x = -\frac{5}{2}$ is not included).

Example 5. Draw the graph of the inequality $y \le 2x$.

Solution : We draw the graph of the equation $y = 2x$. From the equation we get,

x	0	2	–2
y	0	4	–4

In the graph paper, taking the length of the side of a small square as unit, we plot the points (0, 0), (2, 4), (– 2, – 4) and by joining them we draw the line.



Here, the point (1, 0) lies in the lower part the graph-line. At this point
$y - 2x = 0 - 2 \times 1$
$= -2 < 0$. Thus the graph of the given inequality is that part of the plane which is formed by the graph-line and the part below it (that is, the part where the point (1, 0) lies).

Example 6. Determine the solutions of the simultaneous inequalities
$2x - 3y - 1 \ge 0$ and $2x + 3y - 7 \le 0$.

Solution : First we draw the graphs of the two equations,

$$2x - 3y - 1 = 0 \qquad (i)$$

and $$2x + 3y - 7 = 0 \qquad (ii)$$

From (i) we get,

$3y = 2x - 1$ or, $y = \frac{2x - 1}{3}$

Here,

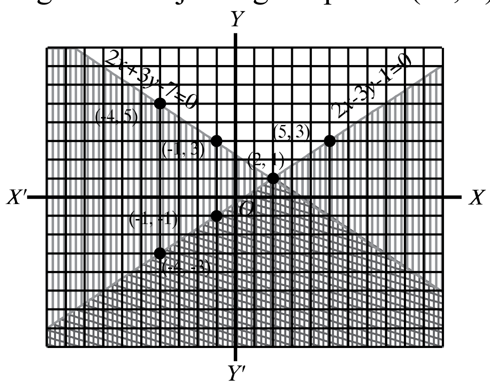
x	5	–4	–1
y	3	–3	–1

From (ii) we get, $3y = -2x + 7$ or, $y = \frac{-2x+7}{3}$

Here,

x	–1	2	–4
y	3	1	5

Now in squared paper, taking the length of the side of a small square as unit, we draw the graph-line of the equation $2x - 3y - 1 = 0$ by ploting and then joining the point (5, 3), (–4, –3), (–1, –1) and then the graph-line of the equation $2x + 3y - 7 = 0$ by plotting and then joining the points (–1, 3), (2, 1), (–4, 5).



At the origin (0, 0) the value of the expression $2x - 3y - 1$ is –1 which is negative. Therefore, $2x - 3y - 1 < 0$ for all points at the origin side of the graph-line of the equation $2x - 3y - 1 = 0$ and $2x - 3y - 1 > 0$ for all points on the other side. Thus the shaded region of the plane below the graph-line along with it is the graph of the inequality $2x - 3y - 1 \geq 0$.

Again at (0, 0) the value of the expression $(2x - 3y - 7)$ is –7 which is negative. Thus $2x + 3y - 7 < 0$ for all points at the origin side of the graph-line of the equation $2x + 3y - 7 = 0$.

Therefore, the shaded region of the plane below the graph-line along with it is the graph of the inequality $2x + 3y - 7 \leq 0$.

Thus the intersection of these two shaded regions along with the related parts of the two graph lines is the graph of the simultaneous solutions of the given two inequalities.

In the figure the dark shaded region (with the boundaries) denotes this graph.

Exercise 6·3

1. Which one is the solution set of the inequality <u>5x</u> + 5 > 25?

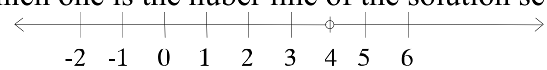
   a. $S = \{x \in R : x > 4\}$
   b. $S = \{x \in R : x < 4\}$
   c. $S = \{x \in R : x \leq 4\}$
   d. $S = \{x \in R : x \geq 4\}$

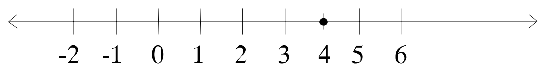
2. For which value of $x + y = -2$ gives $y = 0$?
   a. 2 b. 0 c. 4 d. -2
3. Which are the correct solutions of the equation $2x + y = 3$?
   a. $(1, -1), (2, -1)$ b. $(1, 1), (2, -1)$
   c. $(1, 1), (-2, 1)$ d. $(-1, 1), (2, -1)$

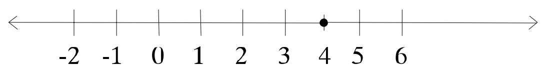
Answer the question nos. 4 & 5 from the inequality given below :

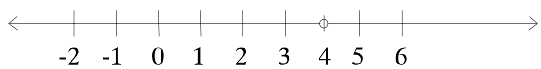
$$x \leq \frac{x}{4} + 3$$

4. Which one is the solution set of the inequality ?
   a. $S = \{x \in R : x > 4\}$ b. $S = \{x \in R : x < 4\}$
   c. $S = \{x \in R : x \leq 4\}$ d. $S = \{x \in R : x \geq 4\}$
5. Which one is the nuber line of the solution set of the above inequality?

a.


b.


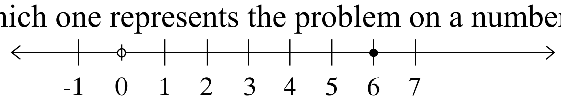
c.


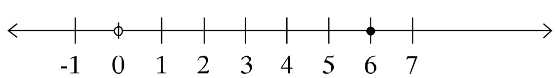
d.


Answer the questions 6 & 7after reading the section given below:

A student has bought x pencils at Tk. 10.00 each and $(x+3)$ khatas at Tk. 6.00 each. If the total price of these does not exceed Tk. 114.00.

6. Which inequality expresses the problem?
   i $10x + 6(x+3) \leq 114$ ii $10x + 6(x+3) \geq 114$
   iii $10x + 6(x+3) < 114$
   Which one of the following is true?
   a. i b. ii c. iii d. i & ii
7. What is the maximum number of pencils the student bought?
   a. 1 b. 3 c. 5 d. 6
8. Which one represents the problem on a number line?

a.


b.


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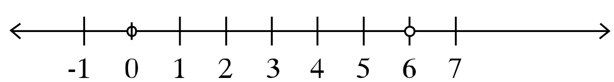

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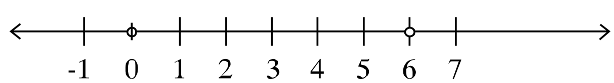


9. Draw the graph of the following inequalities:

(i) $x - y > -10$ (ii) $2x - y < 6$

(iii) $3x - y \geq 0$ (iv) $3x - 2y \leq 12$

(v) $y < -2$ (vi) $x \geq 4$

(vii) $y > x + 2$ (viii) $y < x + 2$

(ix) $y \geq 2x$ (x) $x + 3y < 0$

10. Draw the graph of the solution sets of each pair of the following inequalities:

(i) $x - 3y - 6 < 0$ and $3x + y + 2 < 0$

(ii) $x + y - 4 \leq 0$ and $2x - y - 3 \geq 0$

(iii) $x - y + 3 > 0$ and $2x - y - 6 \geq 0$

(iv) $x + y - 3 > 0$ and $2x - y - 5 > 0$

(v) $x + 2y - 4 > 0$ and $2x - y - 3 > 0$

(vi) $5x + 2y > 11$ and $7x - 2y > 3$

(vii) $3x - 3y > 5$ and $x + 3y \leq 9$

(viii) $5x - 3y - 9 > 0$ and $3x - 2y \geq 5$.

11. The air distance of Singapore from Hazrat Shahjalal airport is 1793 km. The maximum speed of a Bangladesh Biman is 500 km/hour. But on the way from Hazrat Shahjalal airport, it faces air flowing at 60 km/hour from the opposite direction.
    a. Express the problem of stimulus in terms of an inequality taking the required time as t.
    b. Find the required time of non-stop flying from Hazrat Shahjalal airport to Singapore airport using the inequality in (A) and show it on anumber line.
    c. Take x as time of returning from Singapore airport to Hazrat Shahjalal airport then express the problem in the form of an inequality and solve it graphically.
12. Between two numbers, the result of subtraction of 5 times of the second number from 3 times of the first one is grater than 5. Again when 3 times of the second number subtracted from the first one, the result is not more than 9.
    a. Express the conditions stated by stimulus in the form of inequalities.
    b. If 5 times of the first number is less than the sum of twice the first number and 15, express the possible values of the number in the form of an inequality.
    c. Draw the graph of each pair of inequalities obtained in (A).

Chapter Seven

Infinite Series

Sequence and finite series are discussed in detail in the Secondary Algebra Book. But there is a direct relationship between sequence and infinite series. Infinite series can be obtained after assigning an appropriate mathematical symbol to the terms of a sequence. Infinite series are discussed in this chapter.

After Completing the chapter, the students will be able to

- explain the idea of a sequence.
- identify the infinite series
- explain the condition of existing the sum of an infinite geometric series
- sum an infinite series geometric series
- transform a recurring decimal number into an infinite geometric series and express in fraction.

Sequences

We observe the relationship between two sets of numbers given below.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \cdots\cdots & n & \cdots\cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ 1 & 4 & 9 & 16 & 25 & \cdots\cdots & n^2 & \cdots\cdots \end{array}$$

Here every natural number n is related to its square n^2. That is, the set of square numbers $\{1, 4, 9, 16, ...\}$ is obtained for the set of natural numbers $N = \{1, 2, 3, 4, ...\}$ under a certain rule.

This set of arranged sequence numbers is a sequence. Suppose, some numbers are arranged successively under a definite rule such that the relationship between any two successive terms is known. The set of numbers arranged in this way is called sequence.

The relationship shown above is called a function and written as $F(n) = n^2$. The general term of this sequences is n^2. The number of terms of a sequences is infinite. In terms of the general term n^2, the sequence can be written as $\{n^2\}$, $n = 1, 2, 3, \ldots\ldots$, or, $\{n^2\}_{n=1}^{+\infty}$ or, $\{n^2\}$.

The first number of a sequences is called the first term, the second number of a sequences is called the second term and so on. In the above sequence $\{1, 4, 9, 16, \quad ....\}$, first term =1, Second term = 2 and so on.

Four examples of sequences are given below:

$\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$, $\frac{1}{2^4}$, ..., $\frac{1}{2^n}$, ...

3, 1, Ñ1, Ñ3, ..., $(5-2n)$, ...

1, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{4}{7}$, ..., $\frac{n}{2n-1}$, ...

$\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{17}$, ..., $\frac{1}{n^2+1}$, ...

Activity:

1. Find the general term of the following sequences:

   (i) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \cdots\cdots$ (ii) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \cdots\cdots$

   (iii) $\frac{1}{2}, \frac{1}{2}, \frac{3}{2^3}, \frac{4}{2^4}, \cdots\cdots$ (iv) $1, \sqrt{2}, \sqrt{3}, 2, \cdots\cdots.$

2. Write down the sequences from their general terms are given below:

   (i) $1+(-1)^n$ (ii) $1-(-1)^n$ (iii) $1+\left(-\frac{1}{2}\right)^n$ (iv) $\frac{n^2}{\sqrt[n]{\pi}}$ (v) $\frac{\ln n}{n}$

   (vi) $\cos\left(\frac{n\pi}{2}\right)$

3. Every one of you writes the general term of a sequence and then write the sequence.

Series

If the terms of a sequences are connected successively by a '+' sign, there a series is formed. As an example, $1+4+9+16+.......$ is a series, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+........$ is also a series.

The ratio between two successive terms of this series is the same. Therefore, the characteristics of a series depends on the relationship between two consecutive terms of it. Geometric series is an important one among all series.

Depending on the number of terms, series can be divided into two classes:

(i) Finite series and (ii) Infinite series

Finite series is a series with limited terms and infinite series is a series with unlimited terms.

Finite series are discussed in the secondary Algebra book. Infinite series will be discussed here.

Infinite Series

If $u_1, u_2, u_3, ..., u_n, ...$ is a sequence of real numbers, then

$u_1+u_2+u_3+........+u_n+.........$ is called an infinite series of real numbers and u_n is called the n-th term of this series.

Partial Sum of Infinite Series

Let $u_1+u_2+u_3+u_3+.........+u_n+.........$ be an infinite series. Then its:

1st partial sum is $S_1 = u_1$
2nd partial sum is $S_2 = u_1 + u_2$
3rd partial sum is $S_3 = u_1 + u_2 + u_3$
....
....
$\therefore$ The nth partial sum is $S_n = u_1 + u_2 + u_3 + ...u_n$

That is, the nth partial sum of an infinite series is the sum of the first *n* number of terms (where $n \in N$) of the series.

Example: 1. Find the partial some of the following two series:

(a). $1+2+3+4+..........$
(b). $1-1+1-1+............$

Solution: (a) First term of the seris $a = 1$ and common difference $d = 1$. Therefore, the given series is an arithmetic progression.

$$\therefore \text{ Sum } S_n = \frac{n}{2}\{2\cdot 1 + (n-1)\cdot 1\} \qquad [\because S_n = \frac{n}{2}\{2a + (n-1)d\}]$$
$$= \frac{n}{2}\{2 + n - 1\}$$
$$= \frac{n(n+1)}{2}$$

In the above example, putting different values of n we get,

$$S_{10} = \frac{10\times 11}{2} = 55$$
$$S_{1000} = \frac{1000\times 1001}{2} = 500500$$
$$S_{100000} = \frac{100000\times 100001}{2} = 5000050000$$
....
....

Here as *n* becomes large the value of S_n also becomes large. Therefore, the given infinite series has no sum.

Solution: (b) The given series is $1-1+1-1+............$.

1st partial sum is $S_1 = 1$

2nd partial sum is $S_2 = 1 - 1 = 0$

3rd partial sum is $S_3 = 1 - 1 + 1 = 1$

4th partial sum is $S_4 = 1 - 1 + 1 + 1 - 1 = 0$

....
....

In the above example it is seen that, for odd n, the nth partial sum $S_n = 1$ and for even n, the nth partial sum $S_n = 0$.

It is observed that there does not exist such a finite number which can be called the sum of the series.

Sum of Infinite Series in Geometric Progression

$a + ar + ar^2 + ar^3 + ...$ is a geometric series. Its first term is a and common ratio is r. Therefore, the nth term of the series $= ar^{n-1}$, where $n \in N$ and $r \neq 1$; then the nth partial sum of this geometric series is

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

$$= a \cdot \frac{r^n - 1}{r - 1}, \quad \text{when} \quad r > 1$$

$$\text{and} \quad S_n = a \cdot \frac{1 - r^n}{1 - r}, \quad \text{when} \quad r < 1$$

We observe:

(i). Let $|r| < 1$, that is $-1 < r < 1$. In this case if the value of n increases then the value of $|r^n|$ decreases. Thus making n sufficiently large (that is , when $n \to \infty$) the value can be decreased indefinitely, that is, $|r^n|$ approaches 0. From this we can say that, if $|r| < 1$, then the limiting value of r^n is zero and consequently, the limiting value of

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{r^n}{1 - r}$$

$$= \frac{a}{1 - r}$$

Therefore, the sum of the infinite series $a + ar + ar^2 + ...$ is $S_\infty = \dfrac{a}{1 - r}$.

(ii). Let $|r| > 1$, that is $r > 1$ or, $r < -1$. In this case if the value of n increases then the value of $|r^n|$ increases. Thus making n sufficiently large the value of $|r^n|$ can be increased indefinitely. From this it is clear that there does not exist a finite numbers S which can be considered as the limiting value (when n is indefinitely large) of S_n.

Thus, if $|r| > 1$, then the sum of the infinite series does not exist.

(iii). If $r=-1$, then the limiting value of S_n (when n indefinitely large) can not be found, since the value of $(-1)^n$ is oscillatory between the value $(-1)^n=-1$ (when n is odd) and $(-1)^n=1$ (when n is even). In this case the series is

$$a-a+a-a+a-a+......$$

and the sum of the infinite series does not exist.

If $|r|<1$ that is, $-1<r<1$, then the sum of the infinite series $a+ar+ar^2+......$ is $S=\dfrac{a}{1-r}$. For other values of r, the sum of the series does not exist.

Comment: The sum of the infinite geometric series (if exists) is sometimes denoted by S_∞ and it is called the sum of the series up to infinity.

That is, $a+ar+ar^2+ar^3+............$ up to infinity $S_\infty=\dfrac{a}{1-r}$, when $|r|<1$.

Activity: 1. In each case below, the first term a and the common ratio r of an infinite series are given. Write down the series and find the sum if it exists.

(i) $a=4, r=\frac{1}{2}$ (ii) $a=2, r=-\frac{1}{3}$ (iii) $a=\frac{1}{3}, r=3$

(iv) $a=5, r=\frac{1}{10^2}$. (v) $a=1, r=-\frac{2}{7}$ (vi) $a=81, r=-\frac{1}{3}$.

2. Every one of you, write an infinite series.

Example 2. Find the sum (if it exists) of each of the following infinite geometric series.

(1). $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+.........$

(2). $1+0\cdot1+0\cdot01+0\cdot001..........$

(3). $1+\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2\sqrt{2}}+\frac{1}{4}+.........$

Solution: (1). Here $a=\frac{1}{3}$ and common ratio $r=\frac{1}{3^2}\times\frac{3}{1}=\frac{1}{3}<1$

$\therefore$ The sum to infinity of the series is, $S_\infty=\dfrac{a}{1-r}$

$$=\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{3}\times\frac{3}{2}=\frac{1}{2}$$

Solution: (2). Here $a=1$ and common ratio $r=\frac{0\cdot 1}{1}=\frac{1}{10}<1$

$\therefore$ The sum to infinity of the series is, $S_\infty=\frac{a}{1-r}=\frac{1}{1-\frac{1}{10}}=\frac{10}{9}=1\frac{1}{9}$

Solution: (3). Here $a=1$ and common ratio $\text{r}=\frac{\frac{1}{\sqrt{2}}}{1}=\frac{1}{\sqrt{2}}<1$

$\therefore$ The sum to infinity of the series is, $S_\infty=\frac{a}{1-r}=\frac{1}{1-\frac{1}{\sqrt{2}}}$

$$=\frac{\sqrt{2}}{\sqrt{2}-1}=3.414 \text{ (Approx.)}$$

Transformation of Repeating (or Recurring) Decimals in Rational Fractions

Example: 3. Express in rational fractions of the following repeating decimals (a) $0\cdot\dot{5}$ (b) $\cdot\dot{1}\dot{2}$ (c) $1\cdot\dot{2}3\dot{1}$

Solution: (a). $0\cdot\dot{5}=0\cdot 555..........$

$=0\cdot 5+0\cdot 05+0\cdot 005+..........$

It is an infinite geometric series whose first term $a=0\cdot 5$ and common ratio $r=\frac{0\cdot 05}{0\cdot 5}=0.1$

$\therefore 0\cdot\dot{5}=\frac{a}{1-r}=\frac{0\cdot 5}{1-(0\cdot 1)}=\frac{0\cdot 5}{0\cdot 9}=\frac{5}{9}$

Solution: (b). $\cdot\dot{1}\dot{2}=\cdot 12121212..........$

$=\cdot 12+\cdot 0012+\cdot 000012+..........$

It is an infinite geometric series whose first term $a=\cdot 12$ and common ratio $r=\frac{\cdot 0012}{\cdot 12}=\cdot 01$

$\therefore\ \cdot\dot{1}\dot{2}=\frac{a}{1-r}=\frac{\cdot 12}{1-(\cdot 1)}=\frac{\cdot 12}{\cdot 99}=\frac{4}{33}$

Solution: (c). $1\cdot\dot{2}3\dot{1}=1\cdot 231231231..........$

$=1+(\cdot 231+\cdot 000231+\cdot 000000231+..........)$

Here the series in the bracket is an infinite geometric series whose first term $a = \cdot 231$ and common ratio $r = \frac{\cdot 000231}{\cdot 231} = \cdot 001$

$$\therefore\ 1\cdot\dot{2}3\dot{1} = 1 + \frac{a}{1-r}$$

$$= 1 + \frac{\cdot 231}{1-(\cdot 001)} = 1 + \frac{231}{999} = \frac{410}{333}.$$

Exercise 7

1. What is the 12th term 1, 3, 5, 7?
   A. 12 B. 13
   C. 23 D. 25
2. What is the 3rd term of a sequence whose nth term = $\frac{1}{n(n+1)}$?
   A. $\frac{1}{3}$ B. $\frac{1}{6}$
   C. $\frac{1}{12}$ D. $\frac{1}{20}$
3. What is the 20th term of a sequence whose nth term = $\frac{1-(-1)^n}{2}$?
   A. 0 B. 1
   C. -1 D. 2
4. The nth term of a sequence is Un = $\frac{1}{n}$ and Un $<10^{-4}$. The value of n is
   i $n < 10^3$
   ii $n < 10^4$
   iii $n > 10^4$
   Which one is true ?
   A. i & ii B. i & iii
   C. i & iii D. i, ii & iii

Observe the following series and answer the question (5–7) :

$4, \frac{4}{3}, \frac{4}{9}, \ldots\ldots\ldots\ldots$

5. Which one is the 10th term of the series?
   A. $\frac{4}{3^{10}}$ B. $\frac{4}{3^{9}}$
   C. $\frac{4}{3^{11}}$ D. $\frac{4}{3^{12}}$

6. What is the sum of first five terms of the series?

A. $\frac{160}{27}$ B. $\frac{484}{81}$

C. $\frac{12}{9}$ D. $\frac{20}{9}$

7. What is the sum of the series upto infinity?

A. 0 B. 5

C. 6 D. 7

8. Find the 10^{th} term, 15^{th} term and rth term of the given sequences:

(a) 2, 4, 6, 8, 10, 12,...............

(b) $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$,.............

(c) The nth term of the sequence

$$=\frac{1}{n(n+1)},\ n\in N$$

(d) $0, 1, 0, 1, 0, 1,$.............

(e) $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, \frac{5}{81}$,.............

(f) The nth term of the sequence

$$=\frac{1-(-1)^{3n}}{2}.$$

9. The nth term of a sequence is $u_n = \frac{1}{n}$

(a) If $u_n < 10^{-5}$ what will be the value of n ?

(b) If $u_n > 10^{-5}$ what will be the value of n ?

(c) What can be said about the limiting value of u_n (when n is sufficiently large)?

10. Prove by mathematical induction that, the partial sum of n terms of the geometric series

$$a + ar + ar^2 + ar^3 + \text{.............} \text{ is } S_n = a \cdot \frac{1-r^n}{1-r}$$

11. Find the sum of the given series (up to infinity) if exist:

(a) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$.............

(b) $\frac{1}{5}-\frac{2}{5^2}+\frac{4}{5^3}-\frac{8}{5^4}+$.............

(c) $8+2+\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\ldots\ldots\ldots\ldots$

(d) $1+2+4+8+16+\ldots\ldots\ldots\ldots$

(e) $\frac{1}{2}+\left(-\frac{1}{4}\right)+\frac{1}{8}+\left(-\frac{1}{16}\right)+\ldots\ldots\ldots\ldots$

12. Find the sum of first n terms of the series given below :

(a) $7+77+777+\ldots\ldots\ldots\ldots$

(b) $5+55+555+\ldots\ldots\ldots\ldots$

Find also the sum of the series up to infinity.

13. Impose a condition on x under which the infinity series

$\frac{1}{x+1}+\frac{1}{(x+1)^2}+\frac{1}{(x+1)^3}+\ldots\ldots\ldots\ldots$ will have a sum and find that sum.

14. Express each of the given repeating decimals as a rational fraction.

(a) $\cdot\dot{2}\dot{7}$ (b) $2\cdot\dot{3}0\dot{5}$ (c) $\cdot 0\dot{1}\dot{2}3$ (e) $3\cdot 0\dot{4}0\dot{3}$

15. The nth term of a sequence is $U_n = \frac{1}{n(n+1)}$.

a. Find the common ratio after obtaining the series.

b. Find the 15^{th} term and the sum of first 10 terms of the series.

c. Find the sum of the series up to infinity. What is your comment about the limiting value of U_n when n is sufficiently large.

16. Consider the following series:

$\frac{1}{2x+1} + \frac{1}{(2x+1)^2} + \frac{1}{(2x+1)^3} + \ldots\ldots\ldots\ldots\ldots\ldots$

a. Find the series if $x = 1$. What is common ratio of the obtained series?

b. Find the 10^{th} term and the sum of first 10 terms of the series obtained in (a).

c. Find the condition which should be imposed on x, so that the given series will have a sum up to infinity and find the sum.

Chapter Eight

Trigonometry

The word trigonometry derived from two Greek words : Trigonon meaning triangle and metron meaning measurement, Trigonometry is not confined to measurement of triangles and many practical fields of the application ; modern trigonometry is a vibrant branch of mathematics having diverse applications in science, engineering and technology. Trigonometry has two branches ; one is Plane Trigonometry, the other is Spherical Trigonometry. We are concerned with plane trigonometry only.

After completing the chapter, the students will be able to –

- Explain the concept of radian measurment
- Determine the relation between radian measurement and degree measurement
- Indicate trigonometrical ratios and their signs in the quadrant
- Find the trigonometrical ratios of standard angles and associated angles upto 2π
- Find the trigonometrical ratios of angle $-\theta$
- Find the trigonometrical ratios of angle $\left(\frac{n\pi}{2} \pm \theta\right)$ and apply for integer $n(n \leq 4)$
- Solve simple trigonometrical equation.

8.1 Angles in Geometry and in Trigonometry

We choose a point 0 in the plane and draw two mutually perpendicular straight lines passing through 0. It is a standard practice to these as horizontal and vertical lines ; the horizontal line is denoted by XOX' and the vertical line is denoted by YOY'. Each of the regions included between the four right-angles is called a quadrant. The region enclosed by XOY is called the first quadrant, that enclosed by YOX' is called the second quadrant, that enclosed by $X'OY'$ is called the third quadrant and so on (figure 8.1)

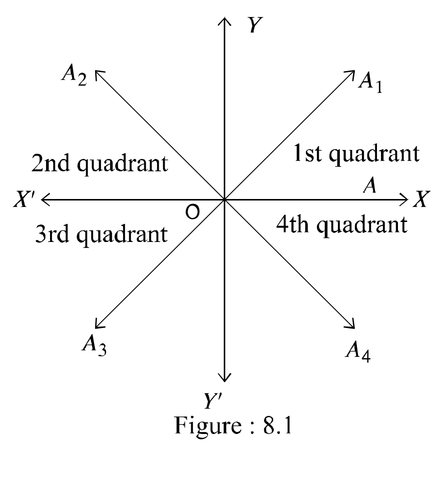


Figure : 8.1

In geometry angle is formed by two rays meeting at a point. In trigonometry an angle is produced by a revolving ray about a fixed ray. OX is taken as the fixed ray. We take a fixed ray OA which can be revolve about OX either in the clockwise or in the anti-clockwise direction. Figure 8.1 show four positions of the revolving ray OA_1, OA_2, OA_3 and OA_4 lying in the first, second, third and fourth quadrant, respectively.

In figure 8.1 there is no indication whether OA has revolved in the clockwise or anti-clockwise direction. In trigonometry a distinction is made between these two modes of revolving of OA. An angle produced by anti-clockwise revolving is considered positive; an angle produced by clockwise revolving is considered negative. In the initial position, the ray OA is coincident with the ray OX. This looked upon as giving the zero angle. If the position OA_1, OA_2, OA_3 and OA_4 are produced by anti-clockwise revolving, then $\angle XOA_1$ is a positive angle having measure between 0^o and 90^o; $\angle XOA_2$ is a positive angle having measure between 90^o and 180^o, XOA_3 is positive angle bying between 180^o and 270^o. 270^o ; lastly $\angle XOA_4$, is a positive angle having measure lying between 270^o and 360^o. But there is no restriction on OA to revolve further in the same direction and thus produce angles of measure greater than 360^o. Each complete revolving produces an angle of 360^o ; as it continues to revolve it will produce positive angle of any measure. Similarly, revolving clockwise, OA will produce negative angles of any measure we may think of. This a fundamental difference between geometrical angles and trigonometrcal angles. In geometry, an angle is always positive and its measure cannot exceed 360^o.

Angle 545^o is positive and greater than six right angle and less than siven right angle. To produce the angle 545^o the ray have to revolve in the anticlock-wise direction upto six right angle or after one full rotation the ray have to revolve move them two right angle and 5^o from the intial position.

Now suppose OA has revolved in the clockwise direction. Then $\angle XOA_4$ is a negative angle having measure between -90^o and 0^o ; $\angle XOA_3$ is a negative angle having measure between -180^o, -90^o $\angle XOA_2$ is a negative angle having measure between -270^o and -180^o ; lastly $\angle XOA_1$ is a negative angle having measure between -360^o and -270^o ;

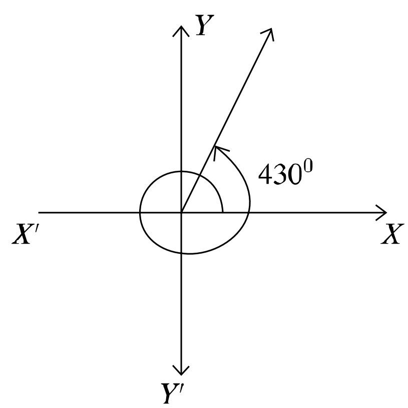


Figure : 8.2

Example 1. In which quadrants do the angles (i) 430° (ii) 545° lie ?

Solution : (i) We have $430^\circ = 360^\circ + 70^\circ$

So the angle 430° is coterminal with the acute angle 70^o, which lies in the first quadrant. So the angle 430° lies in the first quadrant.

(ii) $545^\circ = 545^\circ = 360^\circ + 185^\circ = 6 \times 90^\circ + 5^\circ$

So the angle 545° lies in the third quadrant.

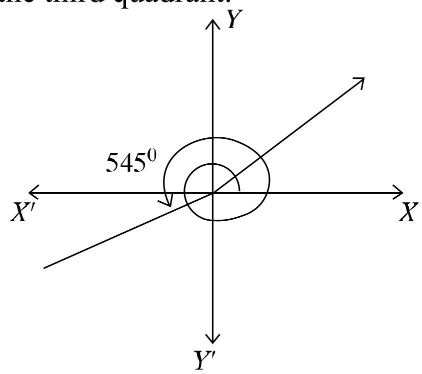


Figure : 8.3

Activity : Determine in which quadrant each of the angles below lie : 330°, 535°, 777°, 1045° ; draw pictures.

Example 2. In which quadrants do the angles $(i) -520^\circ$ (ii) -550° lie ?

Solution : -520^o is negative angle. To produce the angle 520^o any ray have to revolve one full rotation and than rotate more than one right angle and 70^o in the clock-wise direction (fig : 8.4). So the angle lies in the fourth quadrant.

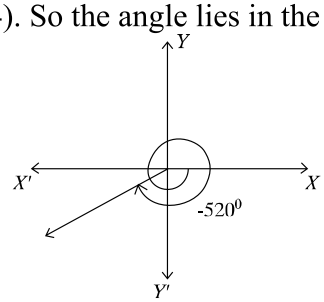


Fig : 8.4

(i) $-520^\circ = -720^\circ = 200^\circ = -2 \times 260^\circ + 200^\circ$

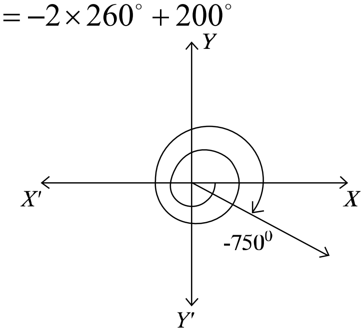


Fig : 8.5

$(ii) -750^\circ = -720^\circ - 30^\circ = -2 \times 360^\circ - 30^\circ$

–750 is negative angle and revolving after two full rotation and revolve 30^o in click-wise rotation it comes in fourth quadrant. So the angle -750° lies in the third quadrant.

Activity : Determine in which quadrant each of the following angle lie : $-100^\circ, -365^\circ, -720^\circ$ and 1320° ; draw pictures.

8.2. Measurement of Angles

To measure any angle we use two unit system :

(i) Sexagesional system

(ii) Circular system

(i) Sexagesional System : In this system right angle is considared as unit of measurement of angle. One degree is one – ninetieth part of a right angle one metre = 1^o. One-minute is one sixtieth part of one degree and that of one-second is one-sixtieth part of one minute. i.e., one second $1''$ = one-second.

i.e., $60'' = 1'$

$60' = 1'$

$90^o = 1$ right angles

Radian : In any circle the angle whch an arc of the circe whose length is equal to the radius of the circle subtends at the centre, is called one radian.

In figure centre of circle PQR is 0 and radian = r, PQ is an arc eqal to the radius. Arc PQ produce angle $\angle POQ$ at centre 0. This measurement of that angle is called radian i.e. $\angle POQ$ an randian.

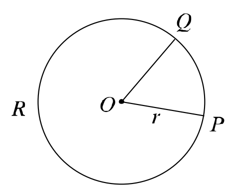


Fig : 8.6

Circular System : In circular system one radian angle is considered as measurement unit of angle. to determine the relation between radian measurement and degree measurement the following proposition need to be known.

Proposition 1 : In any two circles the ratios of the circumferences and the respective diameters and equal.

Proof : We can assume that the circles have the same centre 0 (they are then said to be concentric). Let P , r denote the circumference and radus of the inner (smallr) circle and P, R denote the circumference and radius of the outer (greater) circle (figure 8.7). We choose a natural number $n > 1$ and a point A on the outer circle. Begining with A, we divide the circumference of the outer circle into n equal arcs : joining the dividing points with centre the inner circle also divided into n equal parts. $AB, BO, CB,$.........; joinig A, B, C, D....... with 0, we get n points a, b, c, d, on the circumference of the inner circle ; these point P divide the circumference of the inner circle into n equal arcs. We join the points A, B, C, D....... as well as the points a, b, c, d..... successively. We get consequently (outer circle ABCD .... and inner circle ....) two n sided regular polygons inscribed in the outer and inner circle, respectively.

In the triangle AOB and aob, the angle $\angle AOB = \angle aob$ and $oa = ob$, $OA = OB$ Being isosceles triangles, their other two angles are equal. Therefore ΔAOB are equaiangular, hence similar, Therefore $\frac{AB}{ab} = \frac{ab}{oa} = \frac{R}{r}$.

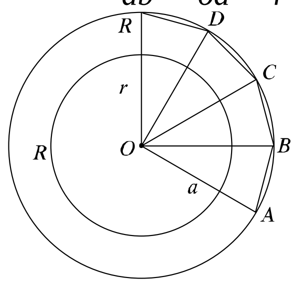


Fig : 8.7

Similarly, $\frac{BC}{bc} = \frac{R}{r}$, $\frac{CD}{cd} = \frac{R}{r}$, etc.

So, $\frac{AB + BC + CD + \text{..........}}{ab + bc + cd} = \frac{R + R + R + \text{......}}{r + r + r + \text{.......}} = \frac{nR}{nr} = \frac{R}{r} = \frac{2R}{2r}$ .... (i)

The result is vey significant; it show that the ratio of the perimeter of the two regular-n-gons (inscribed in the outer and inner circle) is independent of n. Taking n sufficiently large (that is letting $n \to \infty$) the two perimeters can be made as close as we like to the length of the respective circumference. So in the limit, $AB + BC + CD +$.......... = circumtenece of the outer circle = P.

$ab + bc + cd +$.......... = circumtence of the outer circle = p

From (i) $\frac{P}{p} = \frac{2R}{2r} \Rightarrow \frac{P}{2R} = \frac{p}{2r}$

Therefore, $\frac{\text{circumference of the outer circle}}{\text{circumference of the inner circle}} = \frac{\text{diameter of the outer circle}}{\text{diameter of the inner circle}}$

Remark 1 : In any circle the circumference bears a constant ratio to its diameter. This constant ratio is denoted by the Greek letter π (pi). It is an irrational number and expressing in decimal places it will be nonterminating where, π = 3·141592 9535897932 ....

Remark 2 : Generally approximate value of π upto four decimal places is used where π = 3.1416. Using computer the value of π has determined upto millions and millions of digit in decimal place. As we use approximate value of π.

So answer must be approximated. Approximate value of π = 3.1416 will be used as no direction is mentioned.

Corollary : The circumference of any circle of radius r is equal to $2\pi r$.

By Corolary 1, $\frac{\text{Circumference}}{\text{Diametre}}$ = Constant π.

$\therefore$ Circumference = $\pi\times$ diametre

$= \pi\times 2r$

$= 2\times r$.

$\therefore$ Circumference of any cirlce of radius r is $2\pi r$.

Proposition 2 : The centred angle procuced by any arc of a circle is proportional to its arc.

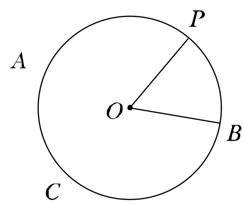


Fig : 8.8

Suppose '0' is the centre and *OB* is radius of circle *ABC*. *P* is another point on the circle. So *BP* is an arc and $\angle POB$ is the centered angle of the circle. So centred angle $\angle POB$ is proportional to arc *BP*. i.e. centred angle $\angle BPQ\infty$ arc *BP*.

Proposition 3 : Prove that radian is a constant angle.

Particular Enunciation : Suppose in the circle ABC of raius r and centre O. $\angle AOB$ is one radian. To prove that $\angle AOB$ is a constant angle.

Construction : Draw perpendicular *OA* upon *OB*.

Proof : *OA* intersect the circumsference at *A*

So, arc AB = one-fourth of the circumference.

$= = \frac{1}{4}\times 2\pi r = \frac{\pi r}{2}$ and]

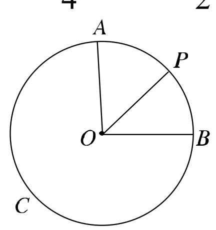


Fig : 8.9

From proposition 2 $\frac{\angle POB}{\angle AOB} = \frac{arc\ \ OB}{Arc\ \ AB}$

$\therefore\ \ \angle POB = \frac{Arc\ PB}{Arc\ AB}\times\angle AOB$

$= \frac{r}{\frac{\pi r}{r}} = \frac{2}{\pi}$ = right angle and π are construct.

So, $\angle POB$ is a constant angle.

Proposition 4 : Any arc of length *s* produce an angle θ in the centre of the circle of radius *r* then $s = r\theta$.

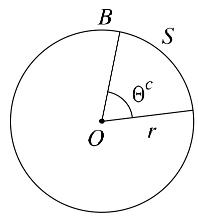


Fig : 8.10

Particular enunciation : Let O is the centre and $OB = r$ is the radius of the circle *ABC*. Arc $AB = s$ and centred angle $\angle AOB = \theta^c$ produced by the arc *AB* then prove that, $s = r\theta$.

Construction : We draw a circle *ABC* taking *O* as centre and *OA* or *OB* as equal radius. Again, draw the arc *BP* equal to radius *OB*. So that it intersect the circle *ABC* at *P*. Join *O*, *P*.

Proof : By construction $\angle POB = 1^c$

We know, centred angle produced by any arc is proportional to arc,

$$\therefore \quad \frac{Arc\ \ AB}{Arc\ \ PB} = \frac{\angle POB}{\angle AOB}$$

or, $\dfrac{s\ \ unit}{r\ \ unit} = \dfrac{\theta^c}{1^c}$

or, $\dfrac{s}{r} = \theta \ \ \therefore \ \ s = r\theta$

8.5 Circular measurement of angle

Definition : The measure of an angle in the radian unit is called its radian measure or circular measure.

Proposition 5 : The radian measure of an angle is equal to the ratio of an arc of any circle subtending that angle at its centre, to its radius.

Suppose $\angle MON$ is a given angle. With centre 0 we draw a circle of suitable radius r; suppose the circle intersects the sides OM, ON of the angle at A and B, respectively (figure 8.10). so constructed angle $\angle AOB$ produced by Arc *AB*. taking an arc *AP* equal to radius *r*.

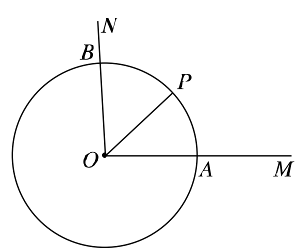


Fig : 8.11

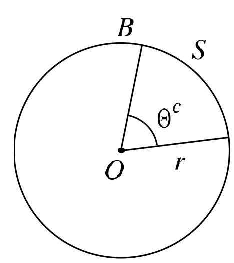


Fig : 8.12

Then $\angle AOP$ = 1 radian

Suppose arc AB = s, from proposition 2.

$$\frac{\angle MON}{\angle AOP} = \frac{arc\ AB}{arc\ AP} = \frac{\text{arc } AB}{\text{Fadius } OA} = \frac{s}{r}$$

$$\therefore\ \angle MON = \frac{s}{r} \times \angle AOP = \frac{s}{r} \times 1 \text{ radian } = \frac{S}{r} \text{ radian}$$

$\therefore$ Circular measure of $\angle MON$ is $\frac{s}{r}$. Where the angle cut are arc *s* centreing its vertex and taking *r* as radius of the circle.

8.6 Relation between the degree and radian measure

We know from proposition 3 (fig 8.9) 1 radian = $\frac{2}{\pi}$ right angle

i.e., $\frac{2}{\pi}$ right angle.

$\therefore$ 1 right angles $\left(\frac{2}{\pi}\right)^c$

or, $90^o = \left(\frac{\pi}{2}\right)^c$

$\therefore\ 1^c = \left(\frac{\pi}{180}\right)^c$ and $1^c = \left(\frac{180}{\pi}\right)^c$

Observation :

(i) 90^o = 1 right angle = $\frac{2}{\pi}$ radian = $\left(\frac{2}{\pi}\right)^c$

i.e., $180^\circ = 2$ right angle = π radian = π^c

If D^o and R^c be the measurment
system then of a angle in sexagessimal and circular

$$\therefore\ D^o = \left(D \times \frac{\pi}{180}\right)^c = R^c$$

So, $D \times \frac{\pi}{180} = R$ (as pure numbers) $\Rightarrow \frac{D}{180} = \frac{R}{\pi}$

From the above discussion the widely use of the relation between degree and radian measures are given below (i) $1^\circ = \left(\frac{\pi}{180}\right)^c$

(ii) $30 = \left(30^\circ \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{6}\right)^c$ (iii) $45 = \left(45^\circ \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{4}\right)^c$

(iv) $60 = \left(60^\circ \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$ (v) $90^\circ = \left(10 \times \frac{\pi}{180}\right)^c = \frac{\pi}{2}$

(vi) $180^\circ = \left(180 \times \frac{\pi}{180}\right)^c = \pi^c$ (vii) $360^\circ = \left(360^\circ \times \frac{\pi}{180}\right)^c = 2\pi^c$

In practical purpose the radian symbol (c) always silent

i.e., $1^\circ = \frac{\pi}{180}$, $30^\circ = \frac{\pi}{6}$, $45^\circ = \frac{\pi}{4}$, $60^\circ = \frac{\pi}{3}$, $90^\circ = \frac{\pi}{2}$

$180^\circ = \pi$, $360^\circ = 2\pi$ etc.

Note 3 : Using the approximate value 3·1416 of π we get

1 degre = $\frac{\pi}{180}$ radian = 10·01745 radian (nearly)

1 redian = $\frac{180}{\pi}$ degrees = 57·29578° (nearly) = 57° 17' 44.81″ (nearly)

We shall use this approsimate value of π in all examples and exercises. When ever an approximate value of π is used the word 'nearly' must be affixed to the result.

Example 3.

(i) Express $30^\circ 12' 36''$. in radians.

(ii) Express $\frac{3\pi}{13}$ in degree, minute and seconds.

Solution :

(i) $30^\circ 12' 36'' = 30^\circ \left(12\frac{36}{60}\right)' = 30^\circ \left(12\frac{3}{5}\right)' = 30^\circ \left(\frac{63}{5}\right)'$

$= \left(30\frac{36}{5\times 60}\right)^\circ = \left(30\frac{21}{100}\right)^\circ = \left(\frac{3021}{100}\right)^\circ$

$= \frac{3021}{100} \times \frac{\pi}{180}$ radian $\left[\because 1^\circ = \frac{\pi^c}{180}\right]$

$= \frac{3021\pi}{18000} = \cdot 5273$ radian [nearly]

$\therefore$ $30^\circ 12' 36'' = \cdot 5273^c$ (nearly)

(ii) $\frac{3\pi}{13} = \frac{3\pi}{13} \times \frac{180}{\pi}$ degree $\left[\because 1^c = \frac{180}{\pi}\right]$

$= \frac{540}{13}$ degree

$= 41^\circ 32' 18 \cdot 46''$.

$\therefore$ $\frac{3\pi}{13}$ radian = $41^\circ 32' \cdot 46''$.

Example 4. The measures of the three angles of a triangle are in the ratio $3:4:5$; Find the circular measures of the angles.

Solution :, The angles are $3x^c, 4x^c$ and $5x^c$ (in any system), where x is a constant. In the circular measure, the sum of the three angles of any triangle is π radian.

Therefore, according to the question $3x^c + 4x^c + 5x^c = \pi^c$

$$\Rightarrow x = \frac{\pi}{12}$$

Therefore the angles are : $3x^c = \left(\frac{3\pi}{12}\right)^c = \left(\frac{\pi}{4}\right)^c = \frac{\pi}{4}$

$$4x^c = \left(\frac{4\pi}{12}\right)^c = \left(\frac{\pi}{3}\right)^c = \frac{\pi}{3}$$

$$5x^c = \left(\frac{5\pi}{12}\right)^c = \frac{5\pi}{12}$$

$\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{12}$ radian.

Example 5. A giant wheel makes 40 revolutions to cover a distance of $1\cdot 75$ kilometer. What is the radius of the wheel ?

Solution : Suppose the radius is r metre. In one revolution the wheel covers a distance equal to its circumference, that is, $2\pi r$ metre.

$\therefore$ In 40 revolutions the wheel covers $40 \times 2\pi r = 80\pi r$ metre.

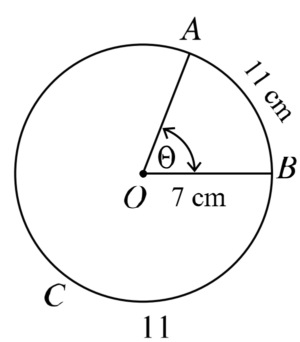
$\therefore$ $80\pi r = 1750$ [because 1.75 km = 1750 metre] [$\because 1km = 1000$ m]

or, $r = \frac{1750}{80\pi}$ metre = $6\cdot 963$ (nearly)

$\therefore$ The radius of the wheel is 6.963 metre (nearly)

Example 6. The arc on the surface of the earth joining Dhaka with Jamalpur subtends an angle of 2° at the centre of the earth. Threating the path as a sphere of radius 6440 kilometre, find the distance between Dhaka and Jamalpur.

Solution : The arc joining Dhaka and Jamalpur is part of the great circle on which these places lie. s be the length in kilometer,



$$2^\circ = 2 \times \frac{\pi}{180} = \frac{\pi}{90} \text{ radian}$$

$$\therefore \quad \frac{\pi}{90} = \frac{s}{r} = \frac{s}{6440}$$

Or, $s = 6640 \times \frac{\pi}{90}$ km. $= \frac{644\pi}{90}$ km. $= 224\cdot 8$ km. (nearly)

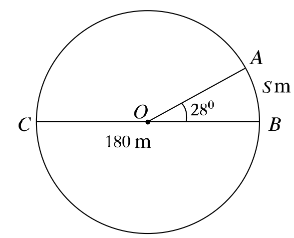
Example 7. Find the circular measure of the angle subtended by an arc of length 11 cm at the centre of a circle radius 7 cm.

Solution : Denoting the angle by θ, its circular measure is $\frac{s}{r}$ radian,

Whre s is the length of the arc and r is the radius of the circle
Here $s = 11$ cm. $r = 7$ cm.

$\therefore \quad \theta = \frac{s}{r} = \frac{11}{7} = 1 \cdot 57$ (nearly)

Ans : $1 \cdot 57$ radian (nearly).



Example 8. The diameter of a circular path is180 metre ; an arc of the circular path subtends an angle of 28° at the dentre. Riding a cicycle Ehsan traverses the arc in 10 seconds. Find the speed of Ehsan.

Solution : Suppose Ehsan's stated from the point *B* of the circle *ABC*. After 7 seconds he comes to the point *A*. So angle of centre produced by the arc *AB* is $\angle AOB = 28^\circ$ and radius $OB = \frac{180}{2}$ = 90 metre

$S = r\theta = 90 \times 28^\circ$ metre

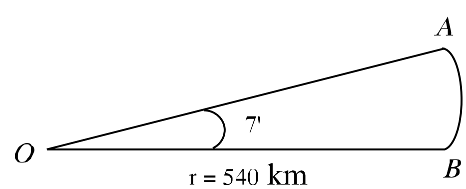
$= 90 \times 28 \times \frac{\pi}{180}$ metre

$= 14\pi$ metre

$= 14 \times 3 \cdot 1416$ metre

$= 43 \cdot 98$ metre (nearly)

$\therefore$ Ehsan's speed $= \frac{43 \cdot 98}{10}$ metre/second = 4.4 metre/second (nearly).



Example 9. A hill subtends an angle of $7'$ at a point 540 kilometre from the foot of the hill. Find the height of the hill.

Solution : We can look upon the hill as an arc of a circle radius 540 km. with centre at the given point O,

$7' = \frac{7^\circ}{60} = \frac{7}{60} \times \frac{\pi}{180}$. Radian. Denoting the height of the hill by s km.

We have $S = r\theta = 540 \times \frac{7\pi}{60 \times 180}$ km

$= \frac{7 \times 3 \cdot 1416}{20}$ km. (nearly)

$= 1 \cdot 1$ km. (nearly)

Answer : The height of the hill is 1100 metre (nearly).

Exercise 8·1

Use $3\cdot1416$ as the approximate value or π in solving problem where necessary. Use of calculator is permitted.

1. (a) Express in radians :

   (i) $75^\circ30'$ (ii) $55^\circ54'53'$ (iii) $33^\circ22'11''$

   (b) Express in degrees :

   (i) $\frac{8x}{13}$ radian (ii) $1\cdot3177$ radian (iii) $0\cdot9759$ radian
2. If we express an angle by D^o and R^c in radian and circular system then prove that $\frac{D}{180}=\frac{R}{\pi}$.
3. The radius of a wheel is 2 metre 2 cm ; find its circumference to four places of decimals.
4. The diameter of a wheel of a car is $0\cdot84$ metre and the wheel makes four revolutions per second. Find the speed of the car.
5. The angle of a triangle are in the ratio $2:5:3$; what are the circular measures of the smallest and the largest angles ?
6. The angles of a triangle are in arithmetical progression and the largest angle is twice the smallest angle. What are the radian measures of the angles ?
7. The arc joining Dhaka with Chittagong subtends an angle of 5° at the centre of the earth. Taking the earth to be a sphere of radius of 6440 km, find the distance of Chittagong from Dhaka.
8. The arc joining Teknaf with Tetulia subtends an angle of $10^\circ6'3''$ at the centre of the earth. Taking the earth to be a sphere of radus 5440 km, find the distance of Tetulia and Teknaf.
9. Riding a bicycle Shahed traverses a segment of a circular path in 11 seconds. The diametr of the circular path is 201 metre and the angle subtended by the segment at the centre is 30^o ; find Shahed's speed.
10. Given that the radius of the earth is 6440 km, what is distance of two places on the surface of the earth which subtend an angle of $32''$ at the cetre of the earth?
11. Express in radian the angle between the munite hand and hour hand of a clock when it is $9\cdot30$ a.m.

    [Hint. One spacing on the dial subtends $\frac{360}{60}$ degree angle at the centre. At $9\cdot30$ A.M. (or P.M.) the hour and minute hands of the clock are $\left(15+2\frac{1}{2}\right)$ or $17\frac{1}{2}$ spacings a part]

12. A person jogging on a circular track traverses a segment of the path in 36 seconds which subtends angle of 60° at the centre. Find the diameter of the circular track.
13. A hill subtends an angle of $8'$ at a point at a distance of 750 kilometre from the foot of the hill. Find the height of the hill.

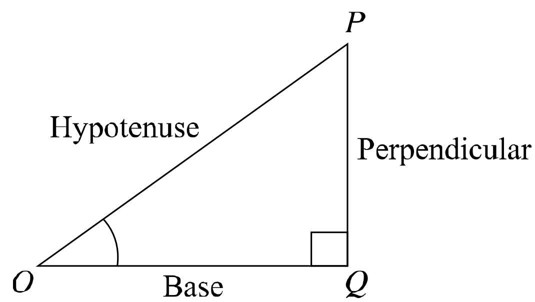
8·7 Trigonometric Ratios

We discuss about trigonometrical ratios of acute angle of this chapter. By the ratios of acute angle, we can determine the technique of any trigonometrical angle. Relation among the ratios and its sign in diffrent quadrant can be explained. Some identity about trigonometrial ratios are to be conceptualized. Trigonometrical ratios and the maximum or minimum values of trigonomtric ratios of standard angle $\left(O, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ are also included here.

(a) Trigonometrial Ratios of Acute Angle

We consider a right angle triagle OPQ to discuss the trigonometrical ratios of acute angle.

$\therefore$ In ΔOPQ, $\angle OPQ$ is right angle OQ base and PQ perpendicular, let $\angle POQ = \theta$.



in ΔOPQ the trigonometrical ratios of acute angle θ (sine, cosine, tangent, secant, cosecant, cotangent) are defined below :

$$\sin\theta = \frac{PQ}{OP} = \frac{\text{Base}}{\text{Hypotenuse}} \qquad \operatorname{cosec}\theta = \frac{OP}{PQ} = \frac{\text{Hypotenuse}}{\text{Pendicular}}$$

$$\cos\theta = \frac{OQ}{OP} = \frac{\text{Base}}{\text{Hypotenuse}} \qquad \sec\theta = \frac{OP}{OQ} = \frac{\text{Hypotenuse}}{\text{Base}}$$

$$\tan\theta = \frac{PQ}{OQ} = \frac{\text{Pendicular}}{\text{Base}} \qquad \cot\theta = \frac{OQ}{PQ} = \frac{\text{Base}}{\text{Pendicular}}$$

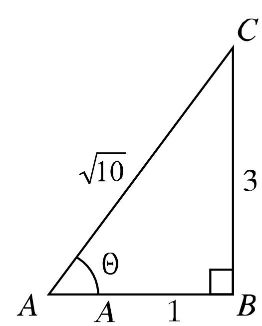
Example 1. In right angle triangle $\tan\theta = 3$ then find the other trigonometrical ratios of θ.

Solution : Let, ABC is a right angle triangle. Hypotenuse = AC, base = AB

perpendicular = BC, and $\angle BAC = \theta$

Here $\tan\theta = 3$

or, $\tan\theta = \frac{\text{Pendicular}}{\text{Base}} = \frac{3}{1}$



$\therefore$ $BC = 3$ unit and $AB = 1$ unit
By Pythagoras
hypotenuse $AC = \sqrt{AB^2 + BC^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$ unite
$\therefore$ other trigonometrical ratios are

$$\sin\theta = \frac{\text{Pendicular}}{\text{Hypotenuse}} = \frac{3}{\sqrt{10}} \quad \cos ine\theta = \frac{\text{Hypotenuse}}{\text{Pendicular}} = \frac{\sqrt{10}}{3}$$

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{1}{\sqrt{10}} \quad \sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{\sqrt{10}}{1} = \sqrt{10}$$

and $\cot\theta = \dfrac{\text{Pendicular}}{\text{Base}} = \dfrac{1}{3}$

Observe : The trigonometrical ratios sine, cosine, tangent, secant, cosecant and cotangent has no unit as ratios is unit base.

> **Activity :** ABC is right angle triangle and $\sin\theta = \dfrac{2}{\sqrt{5}}$. Find the other trigonometrical ratios of θ.

Note : $\sin e\theta = \sin\theta$, $\cos ine\theta = \cos\theta$, $\tan gent\theta = \tan\theta$, $\sec ant\theta = \sec\theta$, $\cos ecant\theta = \cos ec\theta$, $\cot angnt\theta = \cot\theta$

(b) Here we find the trigonometrical ratios of any angle. For this we need to know the standard position of the angle. In cartisian plane right side from origin that is considering the positive direction of x-axis as initial ray be obtained the position of the angle. Here we consider θ as trigonometrical angle and the limit of θ is boundless.

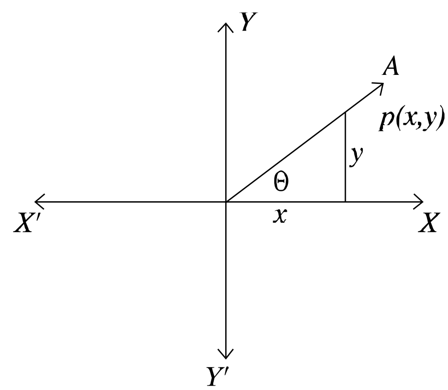


Fig : 8.14

In cartisian plane suppose $X'OX$ as x-axis, $Y'OY$ as y-axis and O as origin. Angle θ is producct by revolving a ray OA in the anticloci-wise direction from x-axis. OX is initial side and OA is terminal side of angle θ. Take a point $P(x, y)$ different form O on the terminal side OA. So perpendicular distance of the point from OX is y and that of x from OY and $\angle OQP$ =1 right angle.

By Pythagoras, hypotenuse = $|OP| = r = \sqrt{x^2 + y^2}$. So any angle θ the trigonometrial ratios will be :

$$\sin\theta = \frac{\text{Pendicular}}{\text{Hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{Pendicular}}{\text{Base}} = \frac{y}{x} \quad [x \neq 0]$$

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{r}{x} \quad [x \neq 0]$$

$$\cos ec\theta = \frac{\text{Hypotenuse}}{\text{Pendicular}} = \frac{r}{y} \quad [y \neq 0]$$

$$\cot\theta = \frac{\text{Base}}{\text{Pendicular}} = \frac{x}{y} \quad [y \neq 0]$$

Obseve 1. As the point P and O are different so $r = |OP| > 0$ and $\sin\theta$, $\cos\theta$ are significant the terminal side OA lie in x-axis then $y = 0$ and in that case $\cos ec\theta$ and $\cot\theta$ is not defined Similarly terminal side lie is y-axis then $x = 0$ and in that case $\sec\theta$ and $\tan\theta$ is not defined.

Observe 2. Take another point $P_1(x_1, y_1)$ different from $P(x, y)$ fig. 8.15(a) and fig. 8.15 (b) on the terminal side OA. Draw perpendicular PM and P_1M_1 from $P(x, y)$ and $P_1(x_1, y_1)$ on x-axis. So, ΔOPM and ΔOP_1M_1 are similar.

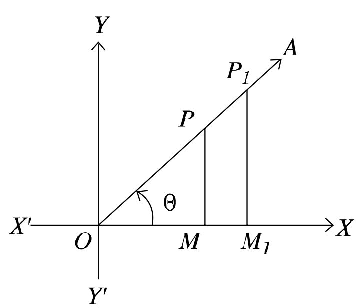


Fig : 15. a

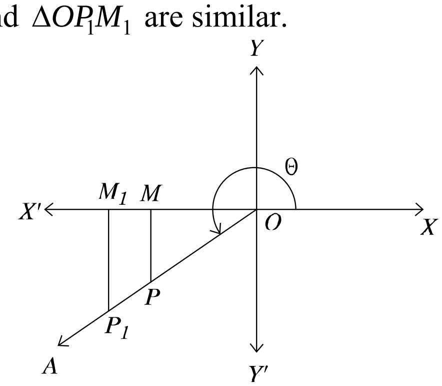


Fig : 15. b

i.e., $\frac{|x|}{|x_1|} = \frac{|y|}{|y_1|} = \frac{|OP|}{|OP_1|} = \frac{r}{r_1}$

Here, $OP = r, OP_1 = r_1$ and x, x_1 and y and y_1 are same sign.

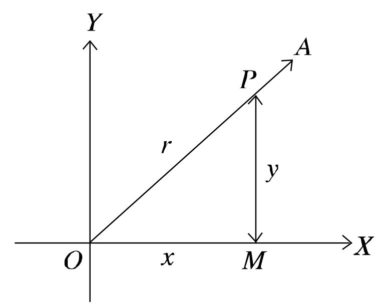
$\therefore \frac{x}{x_1} = \frac{y}{y_1} = \frac{r}{r_1}$ i.e, $\frac{x}{r} = \frac{y}{r}$ and $\frac{x_1}{r_1} = \frac{y_1}{r_1}$

Therefore, $\sin\theta = \frac{y}{r} = \frac{y_1}{r_1}$

$$\cos\theta = \frac{x}{r} = \frac{x_1}{r_1}$$

$$\tan\theta = \frac{y}{x} = \frac{y_1}{x_1} \text{ etc.}$$

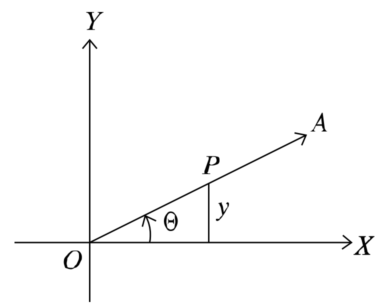
Decision : Value of trigonometrical ratios are not depended upon the points OA.



Observe 3. If θ is acute angle the standard position of OA lies in the first quadrant and $\theta = \angle XOA$ (fig. 8.16). The any point $P(x, y)$ on OA and draw perpendicular PM on OX so that we can find the value of ratios O from the discussion by (a) and (b) $OM = x, PM = y$ and $OP = r$.

(c) Relations of the trigonometrial ratios

From the definations of trigonometrical ratios we see that,



$$\sin\theta = \frac{\text{Base}}{\text{Hypotenuse}},$$

$$\text{cosec}\,\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{1}{\frac{\text{Perpendicular}}{\text{Hypotenuse}}} = \frac{1}{\sin\theta}$$

$$\therefore \quad \sin\theta = \frac{1}{\text{cosec}\,\theta} \quad \text{and } \text{cosec}\,\theta = \frac{1}{\sin\theta}$$

Similarly, $\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}}$, $\sec\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{1}{\frac{\text{Base}}{\text{Hypotenuse}}} = \frac{1}{\cos\theta}$

$$\text{i.e., } \cos\theta = \frac{1}{\sec\theta} \quad \text{and } \sec\theta = \frac{1}{\cos\theta}$$

Similarly, $\tan\theta = \frac{1}{\cot\theta}$ and $\cot\theta = \frac{1}{\tan\theta}$

8.8 Some easy identities concerning about trigonometrical ratios (Identities)

(i) $\sin^2\theta + \cos^2\theta = 1$

Proof : From figure we see that,

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{x}{r}$$

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{y}{r}$$

and $r^2 = x^2 + y^2$

$\therefore \ \sin^2\theta + \cos^2\theta = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$

$\therefore \ \sin^2\theta + \cos^2\theta = 1$ (proved).

From (i) we have, $\sin^2\theta = 1 - \cos^2\theta$ or, $\cos^2\theta = 1 - \sin^2\theta$

Similarly prove that

(ii) $1 + \tan^2\theta = \sec^2\theta$ or, $\sec^2\theta - 1 = \tan^2\theta$

(iii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ or, $\operatorname{cosec}^2\theta - 1 = \cot^2\theta$

Activity : Prove that (by with the help of figure :

(i) $\sec^2\theta - \tan^2\theta = 1$

(ii) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

8.9 Sign of trigonometrical ratios in different quadrants.

In the following figure (fig. 8.18) the cartisian plane are divided into four quadrant by the axis $X'OX$ and $Y'OY$. Namely XOY (1st quadrant), yOX (2nd quadrant) $X'OY'$ (3rd quadrant) and $Y'OX$ (4th quadrant) respectively.

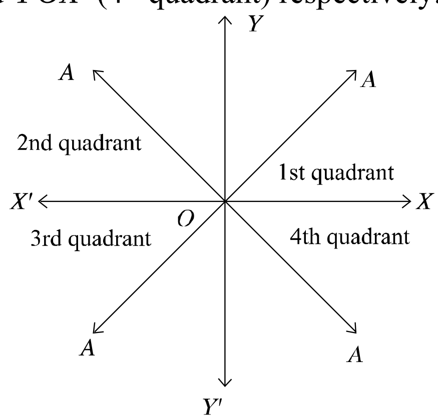


Fig : 8.18

The ray OA rotating in the anticlock-wise direction produce different angle depend the terminal position of OA from the initial position OX . Taking any point $P(x, y)$ on the rotating ray OA. So $|OP| = r$ |. Sign of x and y will be changed depending upon the position of P and the terminal ray OA but r always remain constant.

When OA lie in first quadrant then both x and y are positive so, all trigonometrical ratios in first quadrnat is positive. In and quadrant abscissa x is negative ordinate y is positive. So in 2nd quadrant $\sin\left(\sin\theta = \frac{y}{r}\right)$ 2nd $\operatorname{cosec}\left(\operatorname{cosec}\theta = \frac{r}{y}\right)$ is positive and the other ratios are negative. Similarly in 3rd quadrant abscissa x and ordinate y both are negative and tan $\left(\tan\theta = \frac{-y}{-x} = \frac{y}{x}\right)$ and cot $\left(\cot\theta = \frac{-x}{-y} = \frac{x}{y}\right)$ is positive and the ther ratios are negative . In 4th quadrant abscissa x is positive, ordinate y is negative so cos $\left(\cos\theta = \frac{x}{r}\right)$ and sec $\left(\sec\theta = \frac{r}{x}\right)$ is positive and the other ratios are negative.

Again, in x-axis the value of y in zero, so $\cos ec\left(\cos ec\theta = \frac{r}{y}\right)$ and cot $\left(\cot\theta = \frac{x}{y}\right)$ is not defined. Similarly in y-axis the value of x is zero, so sec $\left(\sec\theta = \frac{r}{x}\right)$ and tan $\left(\tan\theta = \frac{y}{x}\right)$ is not defined. In any position of the point P the ratios sin $\left(\sin\theta = \frac{y}{r}\right)$ and cos $\left(\cos\theta = \frac{x}{r}\right)$ are defined and has real value.

The precise of the above discussion can be shown below with the help of the figure 8.19.

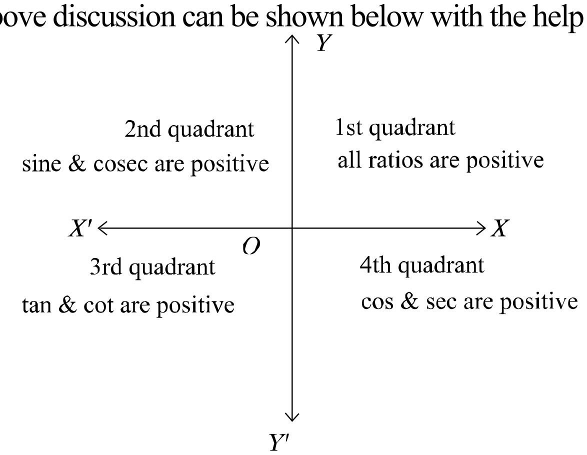


Fig : 8.19

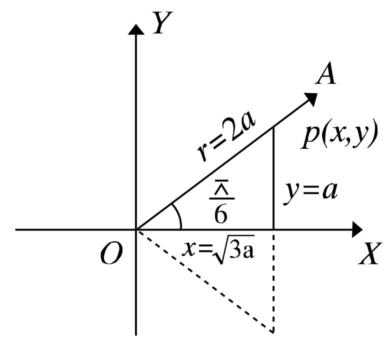
8·10. Trigonometrical raios of standard angles :

We make a table of the widely used angles $\left(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ so that students can easily determined the value of trigonometrical ratios of angle. It is completely discussed in secondary mathematics. So without elaborate discussion we only make a chart or table by the values of ratios.

(a) Trigonometrical ratios of $\frac{\pi}{6}(30^\circ)$.

Here $r = 2a$

$y = a$ and $x = \sqrt{3a}$ and $\angle POB = \dfrac{\pi}{6}$



r=2*a*, *x*=√3a, *y*=*a*

$$\therefore \quad \sin\frac{\pi}{6} = \frac{y}{r} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos\frac{\pi}{6} = \frac{x}{r} = \frac{\sqrt{3a}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{a}{\sqrt{3a}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot\frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3a}}{a} = \sqrt{3}$$

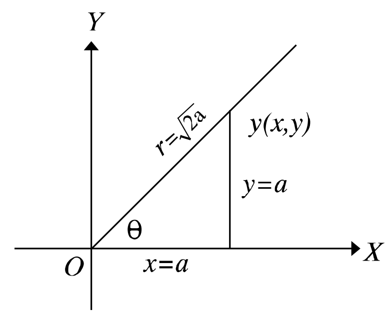
$$\sec\frac{\pi}{6} = \frac{r}{x} = \frac{2a}{\sqrt{5a}} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$$

$$\cos ec\frac{\pi}{6} = \frac{r}{y} = \frac{2a}{a} = 2$$

(b) Trigonometrical ratios of $\dfrac{\pi}{4}(45^\circ)$

Here $r = \sqrt{2a}, x = a$

$y = a$ and $\angle POB = \dfrac{\pi}{6}$



r=√2a, *x*=*a*, *y*=*a*

$$\therefore \quad \sin\frac{\pi}{4} = \frac{y}{r} = \frac{a}{\sqrt{2a}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos\frac{\pi}{4} = \frac{x}{r} = \frac{a}{\sqrt{2a}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan\frac{\pi}{4} = \frac{y}{x} = \frac{a}{a} = 1$$

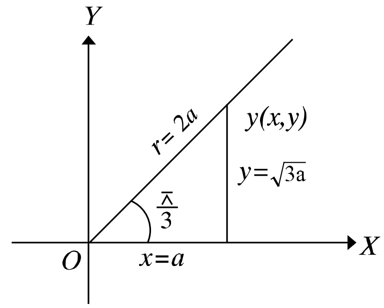
$$\cot\frac{\pi}{4} = \frac{x}{y} = \frac{a}{a} = 1$$

$$\sec\frac{\pi}{4} = \frac{r}{x} = \frac{\sqrt{2a}}{a} = \sqrt{2}$$

$$\cos ec\frac{\pi}{2} = \frac{r}{y} = \frac{\sqrt{2a}}{a} = \sqrt{2}$$

(c) Trigonometrical ratios of $\dfrac{\pi}{3}(60^\circ)$

Here, $x = a$, $y = \sqrt{3a}$, $r = 2a$



$r=2a,\ x=a,\ y=\sqrt{3}a$

and $\angle POB = \frac{\pi}{3}$

$\therefore \quad \sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3a}}{2a} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{x}{r} = \frac{a}{2a} = \frac{1}{2}$

$\tan\frac{\pi}{3} = \frac{y}{x} = \frac{\sqrt{3a}}{a} = \sqrt{3}$

$\cot\frac{\pi}{3} = \frac{x}{y} = \frac{a}{\sqrt{3a}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

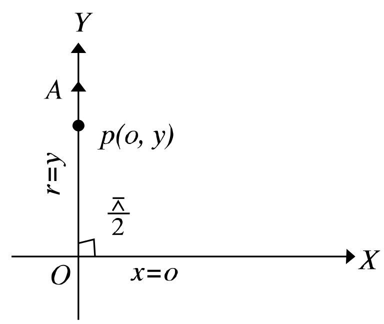
$\sec\frac{\pi}{3} = \frac{r}{x} = \frac{2a}{a} = 2$

$\cos ec\frac{\pi}{3} = \frac{r}{y} = \frac{2a}{\sqrt{3a}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

To determine the value of $\frac{\pi}{2}(90^\circ)$ and $0(0^\circ)$. We use the definition of trigonometrical ratios. Here mentionable that division by zero is not allowed or division by zero is undefined.

(d) Trigonomtrcal ratios of $\frac{\pi}{2}(90^\circ)$: Here the position of the terminal ratios OA is in y-axis. So abscissa an ordinate a point P is 0 and y.

Let, $y = a$ so, $r = a$



$x=o,\ y=a,\ r=y=a$

$\therefore \quad \sin\frac{\pi}{2} = \frac{y}{r} = \frac{a}{a} = 1$

$\cos\frac{\pi}{2} = \frac{x}{r} = \frac{0}{a} = 0$

$\tan\frac{\pi}{2} = \frac{y}{x}\left(=\frac{a}{0}\right)$ undefined i.e., $\tan\frac{\pi}{2}$ undefined

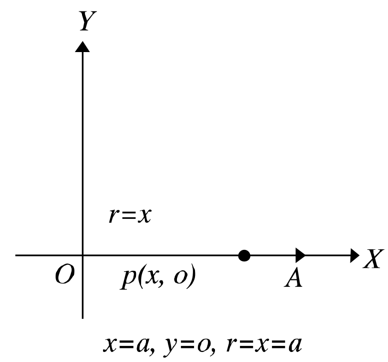
$\cot\frac{\pi}{2} = \frac{x}{y} = \frac{0}{a} = 0$

$\sec\frac{\pi}{2} = \frac{r}{x}\left(=\frac{a}{0}\right)$ undefined i.e., $\sec\frac{\pi}{2}$ undefined

$\cos ec\frac{\pi}{2} = \frac{r}{y} = \frac{a}{a} = 1$

(e) Trigonometrical ratios of O radian (0°)

Draw the terminal ray OA which lie in OX . So the position of OA on OX and the abscissa and ordinate of the point *x* and *0*.



$\therefore \quad \sin 0 = \frac{y}{r} = \frac{0}{a} = 0$

$\cos 0 = \frac{x}{r} = \frac{a}{a} = 1$

$\tan 0 = \frac{y}{x} = \frac{0}{a} = 0$

$\cot 0 = \frac{x}{y}\left(= \frac{a}{0}\right)$ undefined i.e., $\cot 0$ undefined

$\sec 0 = \frac{r}{x} = \frac{a}{a} = 1$

$\operatorname{cosec} 0 = \frac{r}{y}\left(= \frac{a}{0}\right)$ undefined i.e., $\operatorname{cosec} 0$ undefined.

N.B. : If is written $x + y\left(\frac{a}{a}\right)$ only to understand but it is not right. Students directly write undefined.

In the 12th chapter of class ten, the list of trigonometrical ratios $\left(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ are given. For the convinience of the student the list attached below :

Angle	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined
cot	undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0
sec	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undefined
cosec	undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1

8·11 Maasurement and minimum values or range of values of Trigonometric ratios.

In figure 8.20 observe that the ray OA produce an angle θ by the revolving of anticlock-wise direction. We take a point P whose coordinates is $P(x, y)$ on OA the standard position of the angle. So, POQ is right angled triangle and $OP = r$ = by p other use.

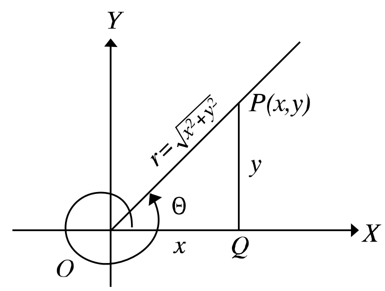


Fig: 8.20

or, $r^2 = x^2 + y^2$

$\therefore$ $x^2 \le r^2$ and $y^2 \le r^2$

or, $|x| \le r$ and $|y| \le r$

or, $-r \le x \le r$ and $-r \le y \le r$

or, $-1 \le \frac{x}{r} \le 1$ and $-1 \le \frac{y}{r} \le 1$

Now POQ is right angle

$$\sin\theta = \frac{x}{r},\ \cos\theta = \frac{y}{r}............(2)$$

$$\operatorname{cosec}\theta = \frac{r}{x},\ \sec\theta = \frac{r}{y}............(3)$$

$$\tan\theta = \frac{y}{x},\ \cot\theta = \frac{x}{y}...........(4)$$

we have from (1) and (2) $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$

Therefore, the value of $\sin\theta$ and $\cos\theta$ is less then -1 and $+1$ is not greater than

From equal (1) and (3) $\operatorname{cosec}\theta \ge 1$ or, $\operatorname{cosec}\theta \le -1$

and $\sec\theta \ge 1$ or, $\sec\theta \le -1$

Therefore, the value of $\sec\theta$ and $\operatorname{cosec}\theta$ is less than -1, $z-1$ and greater than $+1$, +1

as $\tan\theta = \frac{y}{x}$ and $\cot\theta = \frac{x}{y}$

$\therefore$ If abscissa $x = 0$ then $\tan\theta$ is undefined and ordinate $y = 0$ then $\cot\theta$ is underfined concept of undefined (∞) sign and we say.

$-\infty < \tan\theta < +\infty$ Ges $-\infty < \cot\theta < +\infty$.

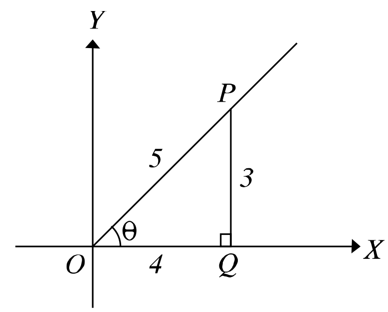
Example 1. Find the trigonometrical ratios where $\cos\theta = \frac{4}{5}$ and θ is acute angle $\left(O < \theta < \frac{\pi}{2}\right)$.

Solution **:** We know, $\cos\theta = \dfrac{\text{base}}{\text{Hypoteneuse}} = \dfrac{4}{5}$ [given]

Here POQ hypotenuse is a right angled triangle

$$PQ = \sqrt{OP^2 - OQ^2} = \sqrt{5^2 - 4^2}$$

$$= \sqrt{25 - 16} = \sqrt{9} = 3 \text{ unite}$$



$$\therefore \quad \sin\theta = \frac{\text{Perpendicular}}{\text{Hypoteneuse}} = \frac{pQ}{OP} = \frac{3}{5}$$

$$\tan\theta = \frac{\text{Hypoteneuse}}{\text{base}} = \frac{pQ}{OQ} = \frac{3}{4}$$

$$\sec\theta = \frac{\text{Perpendicular}}{\text{base}} = \frac{Op}{OQ} = \frac{5}{4}$$

$$\cos ec\theta = \frac{\text{Perpendicular}}{\text{base}} = \frac{Op}{PQ} = \frac{5}{3}$$

$$\cot\theta = \frac{\text{Hypoteneuse}}{\text{Perpendicular}} = \frac{OQ}{PQ} = \frac{4}{3}$$

Alternative : Using trigonometrical ratios

We know, $\sin^2\theta + \cos^2\theta = 1$

or, $\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{4}{5}\right)^2 = 1 - \frac{16}{25}$

$$= \frac{25-16}{25} = \frac{9}{25}$$

$\therefore \ \sin\theta = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$

As θ is acture angle it lie in the first quadrant and all the triagonometrical ratios are postive.

$\therefore \ \sin\theta = \frac{3}{5}$

Here, $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{4}{5}} = \frac{4}{5}$

$$\cos ec\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{4}{5}} = \frac{5}{3}$$

Now from right angled triangle POQ, we get

$$\tan\theta = \frac{\text{perpendicullar}}{\text{base}} = \frac{\text{perpendicullar/hypoenuse}}{\text{base/hypoenuse}} = \frac{PQ/OP}{OQ/OP}$$

$$= \frac{\sin\theta}{\cos\theta} = \frac{3/5}{4/5} = \frac{3}{4}$$

$\cot\theta = \dfrac{\text{base}}{\text{perpendicullar}} = \dfrac{\text{base/hypoenuse}}{\text{perpendicular/hypotenuse}} = \dfrac{OQ/OP}{PQ/OP}$

$= \dfrac{\cos\theta}{\sin\theta} = \dfrac{4/5}{4/5} = \dfrac{4}{3}$

N.B. : $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

By trigonometrical identies $\sec^2\theta - \tan^2\theta = 1$

or, $\tan^2\theta = \sec^2\theta = -1 = \left(\dfrac{5}{4}\right) - 1 = \dfrac{25}{16} - 1 = \dfrac{9}{16}$

$\therefore\ \tan^2\theta = \sqrt{\dfrac{9}{16}} = \dfrac{3}{4}$

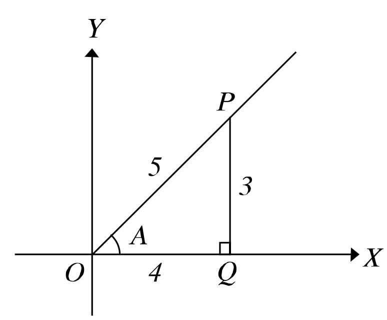
Again $\cos ec^2\theta - \cot^2\theta = 1$

or, $\cot^2\theta = \cos ec^2\theta - 1 = \left(\dfrac{5}{3}\right)^2 - 1 = \dfrac{25}{9} - 1 = \dfrac{16}{9}$

$\therefore\ \cot^2\theta = \sqrt{\dfrac{16}{9}} = \dfrac{4}{3}$

Activity : Find the other trigonometric ratios of acute angle $\theta\left(\dfrac{\pi}{2} < \theta < \pi\right)$ and $\tan\theta = -\dfrac{1}{2}$ with the help of right angle triangle and the trigonometric identities.

Example 2. If $\cos A = \dfrac{4}{5}, \sin B = \dfrac{12}{13}$ and A, B acute angle then find the value of $\dfrac{\tan B - \tan A}{1 + \tan B.\tan A}$.



Solution : Given, $\cos A = \dfrac{4}{5}$

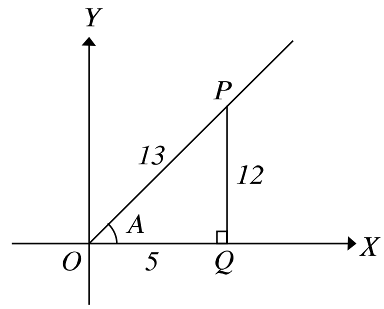
We know, $\sin^2 A = \cos^2 A = 1$

or, $\sin^2 A - \cos^2 A = 1 - \dfrac{16}{25} = \dfrac{9}{25}$.

$\therefore\ \sin A = \sqrt{\dfrac{9}{24}} = \dfrac{3}{5}$ [A acute angle]

$\tan A = \dfrac{\sin A}{\cos A} = \dfrac{\frac{3}{5}}{\frac{4}{5}} = \dfrac{3}{4}$.

Again, $\sin B = \frac{12}{13}$



$\therefore \cos B = \frac{\sqrt{1-\sin^2 B}}{\sqrt{1-\frac{144}{169}}} = \sqrt{\frac{25}{169}}$

$\therefore \cos B = \frac{5}{13}$

$\therefore \tan B = \frac{\sin B}{\cos B} = \frac{12/13}{5/13} = \frac{12}{5}$

Now, $\frac{\tan B - \tan A}{1+\tan B.\tan A} = \frac{\frac{12}{5}-\frac{3}{4}}{1+\frac{12}{5}\cdot\frac{3}{4}}$

$$= \frac{\frac{48-15}{20}}{1+\frac{36}{20}} = \frac{\frac{33}{20}}{\frac{20+36}{20}} = \frac{33}{56}$$

$\therefore \frac{\tan B - \tan A}{1+\tan B.\tan A} = \frac{33}{56}.$

Example 3. Find the value : $\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{4} + \tan^2\frac{\pi}{3} + \cot^2\frac{\pi}{2}$

Solution : We know, $\sin^2\frac{\pi}{6} = \frac{1}{2}$, $\cos^2\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\tan^2\frac{\pi}{3} = \sqrt{3}$ and $\cot^2\frac{\pi}{2} = 0$

$\therefore \sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{4} + \tan^2\frac{\pi}{3} + \cot^2\frac{\pi}{2}$

$= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + (\sqrt{3})^2 + (0)^2$

$= \frac{1}{4} + \frac{2}{4} + 3 \;\; = 3\frac{3}{4}$

Activity : 1. Find the value of $\sin^2\frac{\pi}{4}\cos^2\frac{\pi}{3} + \tan^2\frac{\pi}{6}\sec^2\frac{\pi}{3} + \cot^2\frac{\pi}{3}\cos ec^2\frac{\pi}{4}$.

2. Simplify : $\frac{\sin^2\frac{\pi}{3} + \sin\frac{\pi}{3}\cos\frac{\pi}{3} + \cos^2\frac{\pi}{3}}{\sin\frac{\pi}{3} + \cos\frac{\pi}{3}} - \frac{\sin^2\frac{\pi}{3} - \sin\frac{\pi}{3}\cos\frac{\pi}{3} + \cos^2\frac{\pi}{3}}{\sin\frac{\pi}{3} - \cos\frac{\pi}{3}}$

Example 4 : If $7\sin^2\theta - 3\cos^2\theta = 4$ then prove that, $\tan\theta = \pm\frac{\sqrt{3}}{3}$

Solution : Given, $7\sin^2\theta - 3\cos^2\theta = 4$

or, $7\sin^2\theta + 3(1-\sin^2\theta) = 4$ $[\because \sin^2\theta + \cos^2\theta = 1]$

or, $7\sin^2\theta + 3 - 3\sin^2\theta = 4$

or, $4\sin^2\theta = 1$

or, $\sin^2\theta = \frac{1}{4}$

Again, $\cos^2\theta = 1 - \sin^2\theta$

$= 1 - \frac{1}{4}$

$= \frac{3}{4}$

$\therefore \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$ $\left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$

or, $\tan^2\theta = \frac{\frac{1}{4}}{\frac{3}{4}}$

$\therefore \tan^2\theta = \pm\sqrt{\frac{1}{3}}$

$= \pm\sqrt{\frac{1}{3}}$

$\therefore \tan^2\theta = \pm\sqrt{\frac{3}{3}}$ (proved).

Example 5. If $15\cos^2\theta + 2\sin^2\theta = 7$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ then, find the value $\cot\theta$.

Solution : Given, $15\cos^2\theta + 2\sin^2\theta = 7$

or, $15\cos^2\theta + 2\sin\theta = 7$ $[\because \sin^2\theta + \cos^2\theta = 1]$

or, $15 - 15\sin^2\theta + 2\sin\theta = 7$

or, $15\sin^2\theta - 2\sin\theta - 8 = 0$

or, $(3\sin\theta + 2)(5\sin\theta - 4) = 0$

$\therefore \sin\theta = -\frac{2}{3}$ or, $\sin\theta = \frac{4}{5}$

Both value of $\sin\theta$ are acceptable as $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

If $\sin\theta = -\frac{3}{3}$, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$

If $\sin\theta = \frac{4}{5}$, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

$\therefore \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{\sqrt{5}}{3}}{-\frac{2}{3}} = -\frac{\sqrt{5}}{2}$ $\left[\text{when } \sin\theta = -\frac{2}{3}\right]$

and $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$ $\left[\text{when } \sin\theta = -\frac{4}{5}\right]$

Ans : $\frac{\sqrt{5}}{2}$ or, $\frac{3}{4}$

Example 7. If $A = \frac{\pi}{3}$ and $B = \frac{\pi}{6}$, prove that,

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(ii) $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\cdot\tan B}$

Proof : (i) L.H.S. = $\sin(A+B) = \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \sin\frac{\pi}{2} = 1$

R.H.S. = $\sin A\cos B + \cos A\sin B = \sin\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{1}}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}+\frac{1}{4} = 1$

$\therefore$ L.H.S. = R.H.S. (proved).

Proof : (ii) L.H.S. = $\tan(A-B) = \tan\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$

$= \tan\frac{\pi}{6} - \frac{3}{3}$

R.H.S. = $\frac{\tan A - \tan B}{1+\tan A\cdot\tan B} = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{3}\cdot\tan\frac{\pi}{6}}$

$= \frac{\sqrt{3}-\frac{\sqrt{3}}{3}}{1+\sqrt{3}\cdot\frac{\sqrt{3}}{3}} = \frac{\frac{\sqrt{3}}{3}-\sqrt{3}}{\frac{3}{2}} = \frac{2\sqrt{3}}{3}\times\frac{1}{2} = \frac{\sqrt{3}}{3}$

$\therefore$ L.H.S. = R.H.S. (proved).

Activity : If $A=\frac{\pi}{3}$ and $B=\frac{\pi}{6}$ then prove the identification :

(i) $sin\,(A-B)=\sin\,A\cos B-\cos A\sin B$

(ii) $cos\,(A+B)=cos\,AcosB-sinAsinB$

(iii) $cos\,(A-B)=cos\,A\cos B+sinA\sin B$

(iv) $tan\,(2B)=\frac{2tan}{1-\tan^2 B}\sin\,A\cos B+\cos A\sin B$

Exercise 8·2

1. Find the value of the following by using calculator :

   (i) $\dfrac{\cos\frac{\pi}{4}}{\cos\frac{\pi}{6}+sin\frac{\pi}{3}}$ (ii) $tan\frac{\pi}{4}+tan\frac{\pi}{6}\cdot tan\frac{\pi}{3}$

2. If $cos\theta=-\frac{4}{5}$ and $\pi<\theta<\frac{3\pi}{2}$ then, find the value of $tan\theta$ and $sin\theta$.
3. If $sinA=\frac{2}{\sqrt{5}}$ and $\frac{\pi}{2}<A<\pi$ then, find the value of $cosA$ and $tanA$?
4. Given, $cosA=\frac{1}{2}$ and $cosA$, $sinA$ are of equal sign. Then find value of $sinA$ and $tanA$?
5. Given, $tanA=\frac{5}{12}$ and $tanA$, $cosA$ are of opposite sign. Then find value of $sinA$ and $cosA$.
6. Prove the following identities :

   (i) $tan\,A+cot\,A=secAcosecA$

   (ii) $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}=cosec\theta+\cot\theta=\sqrt{\dfrac{sec\,\theta+1}{sec\theta-1}}$

   (iii) $\sqrt{\dfrac{1-\sin A}{1+sin\,A}}=\sec A-tanA$

   (iv) $sec^4\theta-sec^2\theta=tan^4\theta+tan^2\theta$

   (v) $(sec\theta-cos\theta)(cosec\theta-sin\theta)(tan\theta+\cot\theta)=1$

   (vi) $\dfrac{tan\theta+sec\theta-1}{tan\theta-sec+1}=tan\theta+sec\theta$

7. If $cosecA=\frac{a}{b}$ where $a>b>0$, then prove that, $tanA=\frac{\pm b}{a^2-b^2}$
8. If $cos\theta-sin\theta=\sqrt{2}sin\theta$ then prove that, $cos\theta+sin\theta=\sqrt{2}cos\,\theta$

9. If $tan\theta = \frac{x}{y}(x \neq b)$ then find the value of, $\frac{xsin\theta + ycos\theta}{xsin\theta - ycos + 1}$.

10. If $tan\theta + sec\theta = x$ then find the value of, $sin\theta = \frac{x^2 - 1}{x^2 - y1^2}$

11. If $acos\theta - bsin\theta = c$ then find the value of, $asin\theta + bcos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

12. Find the value of :

(*i*) $sin^2\frac{\pi}{6} + cos^2\frac{\pi}{4} + tan^2\frac{\pi}{3} + cot^2\frac{\pi}{6}$

(*ii*) $3tan^2\frac{\pi}{4} - sin^2\frac{\pi}{3} - \frac{1}{2}cot^2\frac{\pi}{6} + \frac{1}{3}sec^2\frac{\pi}{4}$

(*iii*) $tan^2\frac{\pi}{4} - sin^2\frac{\pi}{3}tan^2\frac{\pi}{6}tan^2\frac{\pi}{3} \cdot cos^2\frac{\pi}{4}$

(*iv*) $\dfrac{tan\frac{\pi}{3} - tan\frac{\pi}{6}}{1 + tan\frac{\pi}{3}tan\frac{\pi}{6}} + cos\frac{\pi}{3}cos\frac{\pi}{6} + sin\frac{\pi}{3}sin\frac{\pi}{6}$

13. Simplify : $\dfrac{1 - sin^2\frac{\pi}{6}}{1 + sin^2\frac{\pi}{4}} \times \dfrac{cos^2\frac{\pi}{3} + cos^2\frac{\pi}{6}}{cosec^2\frac{\pi}{2} - cot^2\frac{\pi}{2}} \div \left(sin\frac{\pi}{3}tan\frac{\pi}{6}\right) + \left(sec^2\frac{\pi}{6} - tan^2\frac{\pi}{6}\right)$

8.12 We discussed the technique about determining the ratios of an acute angle $\left(0 < \theta < \frac{\pi}{2}\right)$.

Some easy identities are proved concerning relations of ratios. Sign of ratios in different quadrants, trigonometrical ratios of standardd angles, idea of maximum and minimum values of ratios are conceptualized. Now we detemine the ratios of negative angle $(-\theta)$. Based on them we discuss the trigonometrical ratios of the angles $\frac{\pi}{2} - \theta, \frac{\pi}{2} + \theta$ $\pi + \theta$, $\pi - \theta$, $\frac{3\pi}{2} + \theta$, $\frac{3\pi}{2} - \theta$, $2\pi + \theta$, $2\pi - \theta$ and $n \times \frac{\pi}{2} + \theta$, $n \times \frac{\pi}{2} - \theta$ where n is positive interger and $\left.0 < \theta < \frac{\pi}{2}\right]$.

8.12 (a) Trigonometrical ratios of $(-\theta)$ where $\left(0 < \theta < \frac{\pi}{2}\right)$

Suppose the revolving ray OA form its initial position OX produce $\angle XOA = \theta$ in anticlockwise direction in 1st qwuadrant and in the same distance produce $\angle XOA' = -\theta$ in clockwise direction (fig. 8.21). Take a point $p(x, y)$ on OA. Draw perpendicular PN on OX from the P. By extanding PN it intersect OA' at Q. So QN is the perpendicular on OX. As $p(x, y)$ is in the 1st quadrant then $x > 0, y > 0$ and $ON = x, PN = y$.

Now form right angled triangle ΔOPN and
ΔOQN, $\angle PON = \angle QON$, $\angle ONP = \angle ONQ$ and ON is common. So the triangles are equal.
$\therefore$ $PN = QN$ and $OP = OQ$.

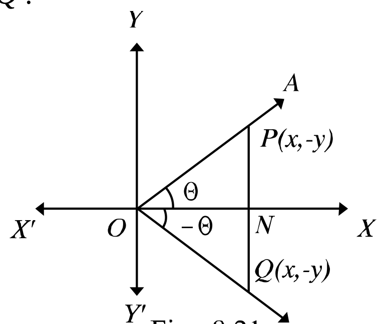


Fig : 8.21

As the point Q is in the 4th quadrant so coordinates of Q is $Q(x, -y)$. In the right angled triangle OQN. ON = base, QN = perpendicular and OQ = hypotensue = r (suppose).

$$\sin(-\theta)\ \frac{\text{parpendicular}}{\text{hypotenuse}} = \frac{QN}{OQ} = \frac{-y}{r} = -\sin\theta$$

$$\cos(-\theta)\ \frac{\text{parpendicular}}{\text{hypotenuse}} = \frac{ON}{OQ} = \frac{x}{r} = \cos\theta$$

$$\tan(-\theta)\ \frac{\text{hypotenuse}}{\text{base}} = \frac{QN}{ON} = \frac{-y}{r} = -\tan\theta$$

Similarly, $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta,\ \sec(-\theta) = \sec\theta,\ \cot(-\theta) = -\cot\theta.$

Example : $\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6},\ \cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4},\ \tan\left(-\frac{\pi}{4}\right) = -\tan\frac{\pi}{4},$

$$\operatorname{cosec}\left(-\frac{\pi}{3}\right) = -\operatorname{cosec}\frac{\pi}{3},\ \sec\left(-\frac{\pi}{3}\right) = \sec\frac{\pi}{3},\ \cot\left(-\frac{\pi}{6}\right) = -\cot\frac{\pi}{6}.$$

8.13 (a). Angle or complimentary angle of $\left(\frac{\pi}{2} - \theta\right)$ wheres $\left(0 < \theta < \frac{\pi}{2}\right)$.

Suppose a revolving ray OA from its initial position OX produce $\angle XOA = \theta$ in the 1st quadrant in anticlock-wise direction. Again another ray OA' from its initial position OX produce $\angle XOY = \frac{\pi}{2}$ in the same direction and then produce $\angle YOA' = -\theta$ in clock-wise direction from the position OY. (figure: 8.21).

So, $\angle XOA' = \frac{\pi}{2} + (-\theta)$

$$= \frac{\pi}{2} - \theta$$

Draw perpendiculars PM and QN from P and Q on OX triangle equal the distance OP and OQ.

Now, $OP = OQ = r$ and P co-ordinate $P(x, y)$.

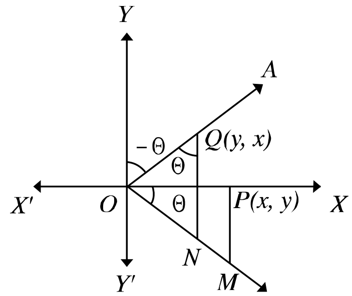


Fig : 8.21

Now, from right angled triangle ΔPOM and ΔQON . $\angle OMP = \angle ONQ$, $\angle POM = \angle OQN$ and $OP = OQ$.

$\therefore$ The triangle are congruent.

$\therefore$ $ON = PM = y$ and Q co-ordinte of $Q(y, x)$.

So from ΔNOQ we get,

$$\sin\left(\frac{\pi}{2} - \theta\right) = \frac{QN}{OQ} = \frac{x}{r} = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{ON}{OQ} = \frac{y}{r} = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{QN}{ON} = \frac{x}{y} = \cot\theta$$

Similarly, $\cos ec\left(\frac{\pi}{2} - \theta\right) = \sec\theta$, $\sec\left(\frac{\pi}{2} - \theta\right) = \cos ec\theta$

and $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$.

Example 9. $\sin\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cos\frac{\pi}{6}$

$$\tan\left(\frac{\pi}{6}\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \cot\frac{\pi}{3}$$

$$\sec\left(\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos ec\frac{\pi}{4}$$

Observe : θ and $\left(\frac{\pi}{2} - \theta\right)$ are complement Angle. Complement of sine, tangent and cotangent are cosine, secant and cotangent, students specialy observe this complements angle

8.13 (b). Trigonometrical ratios of $\left(\frac{\pi}{2}+\theta\right)$ where $\left(0<\theta<\frac{\pi}{2}\right)$.

Suppose revolving ray OA from its initial position OX produce $\angle XOA=\theta$ in the acute $\angle AOA'=\frac{\pi}{2}$ (fig. 8.23) in same director. So, $\angle XOA=\angle YOA'=\theta$ and $\angle XOA'=\frac{\pi}{2}+\theta$.

Let any point $P(x, y)$ on OA, take a point Q on OA' so that $OP=OQ$. Draw perpendicular PM and QN from P and Q on x-axis.

$\therefore\ \angle POM=\angle NQO=\angle YOQ=\theta$.

Now from right angled triangle POM and QON $\quad \angle POM=\angle NQO$

$\angle PMO=\angle QNO$

and $OP=OQ=r$ (let).

$\therefore\ \Delta POM$ and ΔQON congruent.

$\therefore\ |QN|=|OM|=x$

and $|ON|=|PM|=y$

i.e., $ON=-y,\ QN=x$

$\therefore$ Co-ordinate of Q is $Q(-y, x)$

$\therefore$ We get,

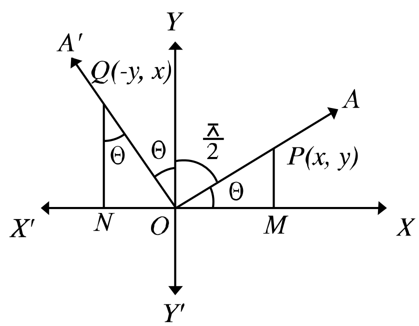


Fig : 8.23

$$\sin\left(\frac{\pi}{2}+\theta\right)=\frac{x}{r}=\cos\theta$$

$$\cos\left(\frac{\pi}{2}+\theta\right)=\frac{-y}{r}=-\sin\theta$$

$$\tan\left(\frac{\pi}{2}+\theta\right)=\frac{x}{-y}=-\cot\theta$$

Similarly,

$\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)=\sec\theta$, $\sec\left(\frac{\pi}{2}+\theta\right)=-\operatorname{cosec}\theta$

and $\cot\left(\frac{\pi}{2}+\theta\right)=-\tan\theta$.

Example 10. $\sin\left(\frac{2\pi}{3}\right)=\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$

$$\cos\left(\frac{3\pi}{4}\right)=\cos\left(\frac{\pi}{2}+\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

$$\tan\left(\frac{5\pi}{6}\right) = \tan\left(\frac{\pi}{2}+\frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = -\frac{1}{\sqrt{3}}$$

Activity : Find the value of $\sec\left(\frac{3\pi}{4}\right)$, $\operatorname{cosec}\left(\frac{5\pi}{6}\right)$ and $\cot\left(\frac{2\pi}{3}\right)$.

8.14 (a). Trigonometical ratios of to $(\pi+\theta)$ where $\left(O<\theta<\frac{\pi}{2}\right)$.

Let the revolving ray OA produce $\angle XOA = \theta$ in the 1st quadrant from the initial line OX in anticlock-wise direction and then produce $\angle AOA' = \pi$ (fig. 8.24) revolving in the same direction.

So, $\angle XOA' = (\pi+\theta)$.

Now take any point $P(x, y)$ on OA. and $Q(-x, y)$ on OA'.

So, that, $OP = OQ = r$ (suppose).

Now from right angled triangles ΔPOM and ΔQON;

$\angle OMP = \angle ONQ$, $\angle POM = \angle QON$ and $OP = OQ = r$.

Therefore the triangles are similar.

$\therefore$ $|PM| = |QN|$ Ges $|OM| = |ON|$

So coordinates of Q is $(-x, -y)$.

$\therefore$ We have,

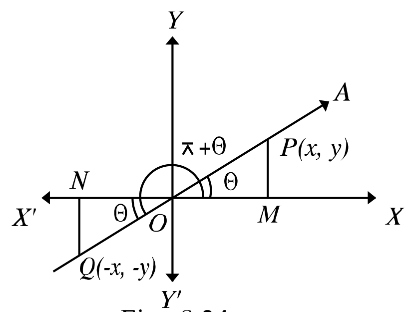


Fig : 8.24

$$\sin(\pi+\theta) = \frac{-y}{r} = -\sin\theta$$

$$\cos(\pi+\theta) = \frac{-x}{r} = -\cos\theta$$

$$\tan(\pi+\theta) = \frac{-y}{-x} = \tan\theta$$

Similarly,

$\operatorname{cosec}(\pi+\theta) = -\operatorname{cosec}\theta$, $\sec(\pi+\theta) = -\sec\theta$ and $\cot(\pi+\theta) = \cot\theta$.

Example 11. $\sin\left(\frac{4\pi}{3}\right) = \sin\left(\pi+\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$$\cos\left(\frac{5\pi}{4}\right) = \cos\left(\pi+\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\tan\left(\frac{7\pi}{6}\right) = \tan\left(\pi+\frac{\pi}{6}\right) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

Activity : Find the value of $\sec\left(\frac{4\pi}{3}\right)$, $\operatorname{cosec}\left(\frac{5\pi}{4}\right)$ and $\cot\left(\frac{7\pi}{6}\right)$.

8.14 (b). Trigonometrical ratios of $(\pi - \theta)$ where $\left(O < \theta < \frac{\pi}{2}\right)$.

Let the revolving ray OA produce $\angle XOA = \theta$ in the 1st quadrant from the initial line OX in anticlock-wise direction and then produce $\angle AOA' = \pi$ (fig. 8.24) revolving in the same direction.

So, $\angle XOA' = \pi + (-\theta) = \pi = \theta$.

Now take a point $P(x, y)$ on OA. Take another point Q on OA'.

So that $OP = OQ = r$.

Draw perpendicular PM and ON on x-axis from P and Q.

In right angled triangle ΔPON and ΔQON ; $\angle OMP = \angle ONQ$, $\angle POM = \angle QON$ and $OP = OQ = r$.

Therefore the triangles are congruent

$\therefore$ $|PM| = |QN|$ and $|OM| = |ON|$

As the coordinates of P is (x, y), so coordinates of Q is $(-x, y)$.

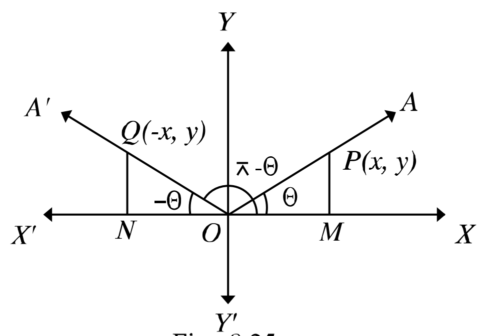


Fig : 8.25

$\therefore$ We have

$$\sin(\pi - \theta) = \frac{y}{r} = \sin\theta$$

$$\cos(\pi - \theta) = \frac{-x}{r} = -\cos\theta$$

$$\tan(\pi - \theta) = \frac{y}{-x} = -\tan\theta$$

Similarly,

$\operatorname{cosec}(\pi - \theta) = \operatorname{cosec}\theta$, $\sec(\pi - \theta) = -\sec\theta$ and $\cot(\pi - \theta) = -\cot\theta$.

Example 12. $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\cos\left(\frac{3\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\tan\left(\frac{5\pi}{6}\right) = \tan\left(\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$

Activity : Find the value of $\operatorname{cosec}\left(\frac{3\pi}{4}\right)$, $\sec\left(\frac{5\pi}{6}\right)$ and $\cot\left(\frac{2\pi}{3}\right)$.

Observe : Q and $(\pi - \theta)$ are supplimentary angle. Sine and cosecant is supplimentary angle are equal and of same in sign. Consiquently cosine and secant, tangent and cotangent also equal and of same sign.

8.15 (a) Trigonometrical ratios of $\left(\frac{3\pi}{2}-\theta\right)$ where $\left(O<\theta<\frac{\pi}{2}\right)$.

From preceding dicussion 8.13 (a) and 8.14 (b) where

$$\sin\left(\frac{3\pi}{2}-\theta\right)=\sin\left\{\pi+\left(\frac{\pi}{2}-\theta\right)\right\}=-\sin\left(\frac{\pi}{2}-\theta\right)=-\cos\theta$$

$$\cos\left(\frac{3\pi}{2}-\theta\right)=\cos\left\{\pi+\left(\frac{\pi}{2}-\theta\right)\right\}=-\cos\left(\frac{\pi}{2}-\theta\right)=-\sin\theta$$

$$\tan\left(\frac{3\pi}{2}-\theta\right)=\tan\left\{\pi+\left(\frac{\pi}{2}-\theta\right)\right\}=\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$$

Similarly

$$\cos ec\left(\frac{3\pi}{2}-\theta\right)=-\sec\theta,\ \ \sec\left(\frac{4\pi}{2}-\theta\right)=-\cos ec\theta$$

and $\cot\left(\frac{3\pi}{2}-\theta\right)=\tan\theta$.

8.15 (b). Trigonometrical ratios of $\left(\frac{3\pi}{2}+\theta\right)$ where $\left(O<\theta<\frac{\pi}{2}\right)$|

From preceding discusstion 8.13 (b) I 8.14 (a) we get

$$\sin\left(\frac{3\pi}{2}+\theta\right)=\sin\left\{\pi+\left(\frac{\pi}{2}+\theta\right)\right\}=-\sin\left(\frac{\pi}{2}+\theta\right)=-\cos\theta$$

$$\cos\left(\frac{3\pi}{2}+\theta\right)=\cos\left\{\pi+\left(\frac{\pi}{2}+\theta\right)\right\}=-\cos\left(\frac{\pi}{2}+\theta\right)=-(-\sin\theta)=\sin\theta$$

$$\tan\left(\frac{3\pi}{2}+\theta\right)=\tan\left\{\pi+\left(\frac{\pi}{2}+\theta\right)\right\}=\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$$

Similarly,

$$\cos ec\left(\frac{3\pi}{2}+\theta\right)=-\sec\theta$$

$$\sec\left(\frac{3\pi}{2}+\theta\right)=\cos ec\theta$$

$$\cot\left(\frac{3\pi}{2}+\theta\right)=-\tan\theta.$$

8.16 (a). Trigonometrical ratios of $(2\pi-\theta)$ where $\left(O<\theta<\frac{\pi}{2}\right)$ Standard position of $(2\pi-\theta)$ is in 4^{th} quadrant and similar to $(-\theta)$. So trigonometrical ratios of $(-\theta)$ and $(2\pi-\theta)$ are equal.

$\therefore\quad \sin(2\pi-\theta)=\sin(-\theta)=-\sin\theta$

$$\cos(2\pi-\theta)=\cos(-\theta)=\cos\theta$$
$$\tan(2\pi-\theta)=\tan(-\theta)=-\tan\theta$$
$$\cos ec(2\pi-\theta)=\cos ec(-\theta)=-\cos ec\theta$$
$$\sec(2\pi-\theta)=\sec(-\theta)=\sec\theta$$
and $\cot(2\pi-\theta)=\cot(-\theta)=-\cot\theta$

8.16 (b) Trigonometrical ratios of $(2\pi+\theta)$ where $\left(O<\theta<\frac{\pi}{2}\right)$ standard position of $(2\pi+\theta)$ is in1st quadrant so the trigonometrical ratios of θ and $(2\pi+\theta)$ are same and equal.
Therefore,

$$\sin(2\pi+\theta)=\sin\theta,\ \cos(2\pi+\theta)=\cos\theta$$
$$\tan(2\pi+\theta)=\tan\theta,\ \cos ec(2\pi+\theta)=\cos ec\theta$$
$$\sec(2\pi+\theta)=\sec\theta,\ \cot(2\pi+\theta)=\cot\theta.$$

8.17. Method of determining the trigonometrical ratios of $\left(n\times\frac{\pi}{2}\pm\theta\right)$ where $\left(O<\theta<\frac{\pi}{2}\right)$

Step 1 : (a) We are to divide the given angle into two parts whose one part is n multiple of $\frac{\pi}{2}$ and the other part is an acute angle i.e., we are to express the given angle in the form $\left(n\times\frac{\pi}{2}\pm\theta\right)$.

Step 2 : If n is even the ratio remain the same that is sine remain sine, cosine remain cosine etc.
If n is odd the ratio will be changed that is, $\sin e$, $\tan gent$ and $\sec ant$ will be change into $\cos ine$, $\cot angent$ and $\cos ecant$, $\cos ine$, $\cot angent$.

Step 3 : After knowing the position in the quadrant of $\left(n\times\frac{\pi}{2}\pm\theta\right)$ we have to put the sign of ratio detrmined in step 2.

Note : 8.17 To determine the ratios students are adviced to follow the method discussed in 8.17.

Example 13. $\sin\left(\frac{9\pi}{2}+\theta\right)$ here n = 9 odd number r. So the ratio sin changed into cos. Again, $\left(9\cdot\frac{\pi}{2}+\theta\right)$ lie in the 10^{th} or 2^{nd} quadrant. So sign of sin is positive.

$\therefore\ \sin\left(\frac{9\pi}{2}+\theta\right)=\cos\theta.$

In case of $\sin\left(\frac{9\pi}{2}-\theta\right)$. $n=9$ is odd and $\left(\frac{9\pi}{2}-\theta\right)$ lie in 9th or 1st quadrant so sign of sign is positive.

$\therefore\ \sin\left(\frac{9\pi}{2}-\theta\right)=\cos\theta$.

In case of $\tan\left(\frac{9\pi}{2}+\theta\right)$, $n=9$ is odd and $\left(\frac{9\pi}{2}+\theta\right)$ lie in the 10th or 2nd quadrant so sign of tan is negative.

$\therefore\ \tan\left(\frac{9\pi}{2}+\theta\right)=-\cot\theta$.

Similarly, $\tan\left(\frac{9\pi}{2}-\theta\right)=\cot\theta$

Activity : Express the angle θ of the ratios $\sin\left(\frac{11\pi}{2}\pm\theta\right)$, $\cos(11\pi\pm\theta)$, $\tan\left(17\frac{\pi}{2}\pm\theta\right)$, $\cot(18\pi\pm\theta)$, $\sec\left(\frac{19\pi}{2}\pm\theta\right)$, and $\operatorname{cosec}(8\pi\pm\theta)$.

8.18. Some example :

Example 14. Find the values of (i) $\sin(10\pi+\theta)$, (ii) $\cos\left(\frac{19\pi}{3}\right)$

(iii) $\tan\left(\frac{11\pi}{6}\right)$, (iv) $\cot\left(\theta-\frac{9\pi}{2}\right)$ I

(v) $\sec\left(-\frac{17\pi}{2}\right)$.

Solution : (i) $\sin(10\pi+\theta) = \sin\left(20\times\frac{\pi}{2}+\theta\right)$

Here, $n=20$ and $\sin\left(20\times\frac{\pi}{2}+\theta\right)$ is in the 21th quadrant or in the first quadrant.

$\therefore\ \sin(10\pi+\theta)=\sin\theta$.

(ii) $\cos\left(\frac{19\pi}{3}\right) = \cos\left(6\pi+\frac{\pi}{3}\right)$

$= \cos\left(12\times\frac{\pi}{2}+\frac{\pi}{3}\right)$ Here $n=12$ and $\frac{19\pi}{3}$ is in first quadrant.

$\therefore\ \cos\left(\frac{19\pi}{3}\right) = \cos\frac{\pi}{3}=\frac{1}{2}$.

$(iii)\ \tan\left(\frac{11\pi}{6}\right) = \tan\left(2\pi - \frac{\pi}{6}\right)$

$= \tan\left(4\times\frac{\pi}{2} - \frac{\pi}{6}\right)$

$= \tan\frac{\pi}{2}$ [$n = 4$ and in the fourth quadrant]

$= -\frac{1}{\sqrt{3}}$.

$(iv)\ \cot\left(\theta - \frac{9\pi}{2}\right) = \cot\left\{-\left(\frac{9\pi}{2} - \theta\right)\right\}$

$= -\cot\left(\frac{9\pi}{2} - \theta\right)$

$= -\cot\left(9\times\frac{\pi}{2} - \theta\right)$

$= -(-\tan\theta)$

$= \tan\theta$ [$n = 9$, $\frac{9\pi}{2} - \theta$ is in the fourth quadrant]

$(v)\ \sec\left(-\frac{17\pi}{2}\right) = \sec\left(\frac{17\pi}{2}\right)$ [$\because \sec(-\theta) = \sec\theta$]

$= \sec\left(17\times\frac{\pi}{2} + 0\right)$

$= \cos ec0$ [$n = 17$, $\frac{17\pi}{2}$ *by* on y-axis] (Undefined)

Example 15. Find the value of :

$(i)\ \sin\frac{11}{90}\pi + \cos\frac{5}{36}\pi + \sin\frac{101}{90}\pi + \cos\frac{31}{30}\pi + \cos\frac{5}{3}\pi$

$(ii)\ \cos^2\frac{\pi}{15} + \cos^2\frac{13\pi}{30} + \cos^2\frac{16\pi}{15} + \cos^2\frac{47\pi}{30}$

Solution : $(i)\ \sin\frac{11}{90}\pi + \cos\frac{5}{36}\pi + \sin\frac{101}{90}\pi + \cos\frac{31}{30}\pi + \cos^{10}\frac{5}{3}\pi$

$= \sin\frac{22\pi}{180} + \cos\frac{25}{180}\pi + \sin\frac{202}{180}\pi + \cos\frac{186}{180}\pi + \cos\frac{300}{180}\pi$

$= \sin\frac{22}{180}\pi + \cos\frac{25}{180}\pi + \sin\left(\pi + \frac{22}{180}\pi\right) + \cos\left(\pi - \frac{25}{180}\pi\right) + \cos\left(2\pi - \frac{60}{180}\pi\right)$

$= \sin\frac{22}{180}\pi + \cos\frac{25}{180}\pi - \sin\frac{22}{180}\pi - \cos\frac{25}{180}\pi + \cos\frac{60}{180}\pi$

$= \cos\frac{\pi}{3}$

$= \frac{1}{2}$.

(ii) $\cos^2\frac{\pi}{15}+\cos^2\frac{13\pi}{30}+\cos^2\frac{16\pi}{15}+\cos^2\frac{47\pi}{30}$

$= \cos^2\frac{2\pi}{30}+\cos^2\frac{13\pi}{30}+\cos^2\frac{32\pi}{30}+\cos^2\frac{47\pi}{30}$

$= \cos^2\frac{2\pi}{30}+\cos^2\frac{13\pi}{30}+\left\{\cos\left(3\cdot\frac{\pi}{2}-\frac{13\pi}{30}\right)\right\}^2+\left\{\cos\left(3\cdot\frac{\pi}{2}+\frac{2}{30}\pi\right)\right\}^2$

$= \cos^2\frac{2\pi}{30}+\cos^2\frac{13}{30}\pi+\left(-\sin\frac{13\pi}{30}\right)^2+\left(-\sin\frac{2\pi}{30}\right)^2$

$= \cos^2\frac{2\pi}{30}+\cos^2\frac{13\pi}{30}+\sin^2\frac{13\pi}{30}+\sin^2\frac{2\pi}{30}$

$= \left(\cos^2\frac{2\pi}{30}+\sin^2\frac{2\pi}{30}\right)+\left(\cos^2\frac{13\pi}{30}+\sin^2\frac{13}{30}\pi\right)=1+1=2$

Example 16. If $\tan\theta=\frac{5}{12}$ and $\cos\theta$ is negative then prove that, $\frac{\sin\theta+\cos(-\theta)}{\sec(-\theta)+\tan\theta}=\frac{51}{26}$

Proof : $\tan\theta=\frac{5}{12}$ and $\cos\theta$ is negative, so angle θ lie in the third quadrant.

i.e., $\tan\theta=\frac{5}{12}=\frac{-5}{-12}=\frac{y}{x}$

$\therefore\ x=-12,\ y=-5$

$\therefore\ r=\sqrt{x^2+y^2}=\sqrt{(-12)^2+(-5)^2}=\sqrt{144+25}$

$=\sqrt{169}=13$

$\therefore\ \sin\theta=\frac{-y}{r}=-\frac{5}{13}$

$\cos\theta=\frac{-x}{r}=-\frac{-12}{13}$ Or, $\sec\theta=\frac{1}{\cos\theta}=-\frac{13}{12}$

$\therefore\ \frac{\sin\theta+\cos(-\theta)}{\sec(-\theta)+\tan\theta}=\frac{\sin\theta+\cos\theta}{\sec\theta+\tan\theta}$ $\left[\because\ \frac{\sin(-\theta)+\cos\theta}{\sec(-\theta)+\sec\theta}\right]$

$=\frac{-\frac{5}{13}-\frac{12}{13}}{-\frac{12}{13}-\frac{5}{13}}=\frac{-\frac{17}{13}}{-\frac{8}{12}}=\frac{17}{13}\times\frac{12}{8}=\frac{51}{26}$ [proved].

Example 17. If $\theta = \frac{\pi}{6}$ then justify the following identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

Proof : Given, $\theta = \frac{\pi}{6}$

$$\therefore \quad \cos 2\theta = \cos\left(2\cdot\frac{\pi}{6}\right) = \cos\frac{\pi}{3} = \frac{1}{2} \quad \left[\cos\frac{\pi}{3} = \frac{1}{2}\right]$$

$$\cos^2\theta - \sin^2\theta = \cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \qquad \left[\because \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \;\; \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}.$$

$$2\cos^2\theta - 1 = 2\cos^2\frac{\pi}{6} - 1 = 2\cdot\left(\frac{\sqrt{3}}{2}\right)^2 - 1$$

$$= 2\cdot\frac{3}{4} - 1 = \frac{3}{2} - 1 = \frac{3-2}{2} = \frac{1}{2}$$

$$2\sin^2 A = 1 - 2\cdot\sin^2\frac{\pi}{6} = 1 - 2\cdot\left(\frac{1}{2}\right)^2$$

$$= 1 - 2\cdot\frac{1}{4} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\frac{\pi}{6}}{1+\tan^2\frac{\pi}{6}}$$

$$= \frac{1-\left(\frac{1}{\sqrt{3}}\right)^2}{1+\left(\frac{1}{\sqrt{3}}\right)^2} \qquad \left[\because \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}\right]$$

$$= \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{\frac{2}{3}}{\frac{4}{3}}$$

$$= \frac{2}{3} \times \frac{4}{3} = \frac{1}{2}$$

$\therefore\ \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$ [justified]

[**N.B. :** The above identities are true for any value of θ. In the next phase the identities will be proved]

Example 18. If $\tan\theta = -\sqrt{3}$, $\frac{\pi}{2} < \theta < 2\pi$ then find the values θ ?

Solution : As $\tan\theta$ is negative, so θ will lie in the 2nd or 4th quadrant

In the 2nd quadrant $\tan\theta = -\sqrt{3} = \tan\left(\pi - \frac{\pi}{3}\right)$

$$= \tan\frac{2\pi}{3}$$

$$\therefore\ \theta = \frac{2\pi}{3}$$

It is not acceptable because $\frac{\pi}{2} < \theta < 2\pi$.

Again, in the 4th quadrant $\tan\theta = -\sqrt{3} = \tan\left(2\pi - \frac{\pi}{3}\right)$

$$= \tan\frac{5\pi}{3}$$

$\therefore\ \theta = \frac{5\pi}{3}$ which satisfy the condition $\frac{\pi}{2} < \theta < 2\pi$

$\therefore$ the value of θ, $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

Example 19. Solve $\left(O < \theta < \frac{\pi}{2}\right)$:

(*i*) $\sin\theta + \cos\theta = \sqrt{2}$

(*ii*) $\sec\theta + \tan\theta = \sqrt{3}$

Solution : (*i*) $\sin\theta + \cos\theta = \sqrt{2}$

or, $\sin\theta = \sqrt{2} - \cos\theta$

or, $\sin^2\theta = 2 - 2\sqrt{2}\cos\theta + \cos^2\theta$

or, $1 - \cos^2\theta = 2 - 2\sqrt{2}\cos\theta + \cos^2\theta$

or, $2\cos^2\theta - 2\sqrt{2}\cos\theta + 1 = 0$

or, $\left(\sqrt{2}\cos\theta - 1\right)^2 = 0$

or, $\sqrt{2}\cos\theta - 1 = 0$

or, $\cos = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$

$\therefore \quad \theta = \dfrac{\pi}{4}$.

$\therefore$ The required solution is $\theta = \dfrac{\pi}{4}$.

(ii) $\sec\theta + \tan\theta = \sqrt{3}$

or, $\dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \sqrt{3}$

or, $\dfrac{1+\sin\theta}{\cos\theta} = \sqrt{3}$

or, $(1+\sin\theta)^2 = (\sqrt{3}\cos\theta)^2$

or, $1+2\sin\theta+\sin^2\theta = 3\cos^2\theta$

or, $(1+2\sin\theta+\sin^2\theta = 3(1-\sin^2\theta)$

or, $1+2\sin\theta+\sin^2\theta+3\sin^2\theta = 3$

or, $4\sin^2\theta+2\sin\theta = 2$

or, $2\sin^2\theta+\sin\theta-1=0$

or, $2\sin^2\theta+2\sin\theta-\sin\theta-1=0$

or, $(2\sin\theta-1)(\sin\theta+1)=0$

$\therefore \quad 2\sin\theta-1=0$ or, $\sin\theta+1=0$

i.e., $\sin\theta = \dfrac{1}{2}$ or, $\sin\theta = -1$

For $O < \theta < \dfrac{\pi}{2}, \sin\theta = -1$ is not acceptable

$\therefore \quad \sin\theta = \dfrac{1}{2} = \sin\dfrac{\pi}{6}$

$\therefore \quad \theta = \dfrac{\pi}{6}$ is the required solution.

Answer : $\theta = \dfrac{\pi}{6}$

N.B. : If $O < \theta < 2\pi$ then $\sin\theta = \dfrac{1}{2}$ and $\sin\theta = -1$ are acceptable on this case solution will be $\theta = \dfrac{\pi}{6}$ or $\theta = \dfrac{3\pi}{2}$.

Example 20. If $O < \theta < 2\pi$ then find the solution of the following equation :

(i) $\sin^2\theta - \cos^2\theta = \cos\theta$

(ii) $2(\sin\theta\cos\theta + \sqrt{3}) = \sqrt{3}\cos\theta + 4\sin\theta$

Solution : (i) $\sin^2\theta - \cos^2\theta = \cos\theta$

or, $1 - \cos^2\theta - \cos^2\theta = \cos\theta$

or, $1 - 2\cos^2\theta - \cos\theta = 0$

or, $2\cos^2\theta + \cos\theta - 1 = 0$

or, $(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\therefore\ 2\cos\theta - 1 = 0$ or, $\cos\theta + 1 = 0$

i.e., $2\cos\theta = \frac{1}{2}$ or, $\cos\theta = -1$

i.e., $\cos\theta = \cos\frac{\pi}{3}$ or, $\cos\theta = \cos\pi$

$\therefore\ \theta = \frac{\pi}{3},\ \pi.$

as $O < \theta < 2\pi$ both the values are acceptable

The required solution : $\theta = \frac{\pi}{3},\ \pi.$

(ii) $2(\sin\theta\cos\theta + \sqrt{3}) = \sqrt{3}\cos\theta + 4\sin\theta$

or, $4(\sin^2\theta\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 3) = 3\cos^2\theta + 8\sqrt{3}\cos\theta\sin\theta + 16\sin^2\theta$

or, $4\sin^2\theta\cos^2\theta + \cancel{8\sqrt{3}\sin\theta\cos\theta} + 12 = 3\cos^2\theta + \cancel{8\sqrt{3}\cos\theta\sin\theta} + 16\sin^2\theta$

or, $4\sin^2\theta\cos^2\theta + 12 = 3\cos^2\theta + 16\sin^2\theta$

or, $4\sin^2\theta(1 - \sin^2\theta) + 12 = 3(1 - \sin^2\theta) + 16\sin^2\theta$

or, $4\sin^2\theta - 4\sin^4\theta + 12 = 3 - 3\sin^2\theta + 16\sin^2\theta$

$4\sin^4\theta + 9\sin^2\theta - 9 = 0$

or, $4\sin^4\theta + 12\sin^2\theta - 3\sin^2\theta - 9 = 0$

or, $(4\sin^2\theta - 3)(\sin^2\theta + 3) = 0$

$\therefore\ 4\sin^2\theta - 3 = 0$ or, $\sin^2\theta + 3 = 0$

i.e., $4\sin^2\theta = 3$ i.e., $\sin^2\theta = -3$

$\therefore\ \sin^2\theta = \frac{3}{4}$ [$\sin^2\theta = -3$ is not acceptable because]

or, $\sin\theta = \pm\frac{\sqrt{3}}{2}$

$\therefore\ \sin\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = -\frac{\sqrt{3}}{2}$

i.e., $\sin\theta = \sin\frac{\pi}{3}$ and $\sin\theta = \sin\left(\pi + \frac{\pi}{3}\right)$

$\therefore \quad \theta = \frac{\pi}{3}$ and $\theta = \frac{4\pi}{3}$

$\therefore$ The required solution $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

Exercise 8·3

1. If $\sin A \frac{1}{\sqrt{2}}$ then find the value of $\sin 2A$

   (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) 1 (d) $\sqrt{2}$

2. In which quadrant the angle 300^o lie ?

   (a) First (b) Second (c) Third (d) Fourth

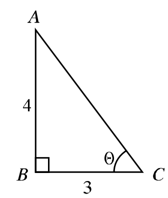
3. If $\sin\theta + \cos\theta = 1$ then find the value of

   (i) 0^o (ii) 30^o (iii) 90^o

Which is correct of the following

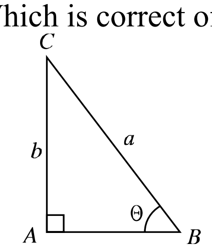
   (a) i (b) ii (c) i and ii (d) i and iii

4. From the figure –



   (i) $\tan\theta = \frac{4}{3}$ (ii) $\sin\theta = \frac{5}{3}$ (iii) $\cos^2\theta = \frac{9}{25}$

Which is correct of the following –



   (a) i and ii (b) i and iii

   (c) ii and iii (d) i, ii and iii

5. $\sin B + \cos C =$ solve ?

   (a) $\frac{2b}{a}$ (b) $\frac{2a}{b}$

   (c) $\frac{a^2+b^2}{ab}$ (d) $\frac{ab}{a^2+b^2}$

6. Which is the value of $\tan B$?

   (a) $\frac{a}{a^2+b^2}$ (b) $\frac{b}{a^2+b^2}$

   (c) $\frac{a}{\sqrt{a^2+b^2}}$ (d) $\frac{b}{\sqrt{a^2+b^2}}$

7. Find the value of

(*i*) $\sin 7\pi$ (*ii*) $\cos\frac{11\pi}{2}$

(*iii*) $\cot 11\pi$ (*iv*) $\tan\left(-\frac{23\pi}{6}\right)$

(*v*) $\operatorname{cosec}\frac{19\pi}{3}$ (*vi*) $\sec\left(-\frac{25\pi}{2}\right)$

(*vii*) $\sin\frac{31\pi}{6}$ (*viii*) $\cos\left(-\frac{25\pi}{6}\right)$

8. Prove that,

(*i*) $\cos\frac{17\pi}{10}+\cos\frac{13\pi}{10}+\cos\frac{9\pi}{10}+\cos\frac{\pi}{10}=0$

(*ii*) $\tan\frac{\pi}{12}\ \tan\frac{5\pi}{12}\ \tan\frac{7\pi}{12}\ \tan\frac{11\pi}{12}=1$

(*iii*) $\sin^2\frac{\pi}{7}+\sin^2\frac{5\pi}{14}+\sin^2\frac{8\pi}{7}+\sin^2\frac{9\pi}{14}=2$

(*iv*) $\sin\frac{7\pi}{3}\cos\frac{13\pi}{6}-\cos\frac{5\pi}{3}\sin\frac{11\pi}{6}=1$

(*v*) $\sin\frac{13\pi}{3}\cos\frac{13}{6}\pi-\sin\frac{11\pi}{6}\cos\left(-\frac{5\pi}{3}\right)=1$

(*vi*) If $\tan\theta=\frac{3}{4}$ and $\sin\theta$ is negative then prove that, $\frac{\sin\theta+\cos\theta}{\sec\theta+\tan\theta}=\frac{14}{5}$.

9. Find the value of :

(*i*) $\cos\frac{9\pi}{4}+\cos\frac{5\pi}{4}+\sin\frac{31\pi}{36}-\sin\frac{5\pi}{36}$

(*ii*) $\cot\frac{\pi}{15}\ \cot\frac{3\pi}{20}\ \cot\frac{5\pi}{20}\ \cot\frac{7\pi}{20}\ \cot\frac{9\pi}{20}$

(*iii*) $\sin^2\frac{\pi}{4}+\sin^2\frac{3\pi}{4}+\sin^2\frac{5\pi}{4}+\sin^2\frac{7\pi}{4}$

(*iv*) $\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\frac{5\pi}{8}+\cos^2\frac{7\pi}{8}$

(*v*) $\sin^2\frac{17\pi}{18}+\sin^2\frac{5\pi}{18}+\cos^2\frac{37\pi}{18}+\cos^2\frac{5\pi}{8}$

10. If $0.\theta=\frac{\pi}{3}$ then prove the following identities :

(*i*) $\sin\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

(*ii*) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(*iii*) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(*iv*) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

11. Find the value of α (alpha) satisfying the given conditions :

(*i*) $\cot\alpha = -\sqrt{3}$; $\dfrac{3\pi}{\alpha} < \alpha < 2\pi$

(*ii*) $\cos\alpha = -\dfrac{1}{2}$; $\dfrac{\pi}{2} < \alpha < \dfrac{3\pi}{2}$

(*iii*) $\sin\alpha = -\dfrac{\sqrt{3}}{2}$; $\dfrac{\pi}{2} < \alpha < \dfrac{3\pi}{2}$

(*iv*) $\cot\alpha = -1$; $\pi < \alpha < 2\pi$

12. Solve : (when $O < \theta < \dfrac{\pi}{2}$)

(*i*) $2\cos^2\theta = 1 + 2\sin^2\theta$

(*ii*) $2\sin^2\theta - 3\cos\theta = 0$

(*iii*) $6\sin^2\theta - 11\sin\theta + 4 = 0$

(*iv*) $\tan\theta + \cot\theta = \dfrac{4}{\sqrt{3}}$

(*v*) $2\sin^2\theta + 3\cos\theta = 3$

13. Solve : (when $O < \theta < 2\pi$)

(*i*) $2\sin^2\theta + 3\cos\theta = 0$

(*ii*) $4(\cos^2\theta + \sin\theta) = 5$

(*iii*) $\cot^2\theta + \cos ec^2\theta = 3$

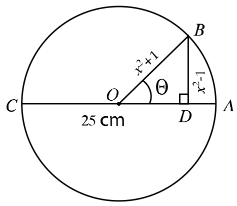
(*iv*) $\tan^2\theta + \cot^2\theta = 2$

(*v*) $\sec^2\theta + \tan^2\theta = \dfrac{5}{3}$

(*vi*) $5\cos ec^2\theta - 7\cot\theta\cos ec\theta - 2 = 0$

(*vii*) $2\sin x\cos x = \sin x\ (O \le x \le 2\pi)$.

14.

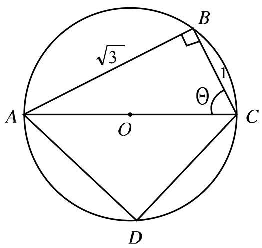


(a) In figure *ABC* is a circluar wheel and length of arc *AB* is 25 cm. then find the value of θ.

(b) What is the speed of the wheel if it revolve five times in a second.

(c) In the figure $\angle BOD = \theta$ then prove that $\tan\theta + \sec\theta = x$ using the value f singθ.

15.



(a) In the figure, *O* is the centre of the circle then find the circular value of *B* and the value of *AC*.

(b) Prove that $\tan A + \tan B + \tan C + \tan dD = O$.

(c) $\sec\theta + \cos\theta = P$ then find the value of *P* and sovle the equation.

Chapter Nine

Exponential & Logarithmic Functions

Exponential functions occur in many context, like population growth, compound interest. With the advent of calculations, use of tables of logarithms and antilogarithms have become redundant. But exponential and logarithmic functions retain their place and importance in the theory and applications of mathematics.

After completing the chapter, the student will be able to –

- Explain, rational and irrational exponents
- Prove and apply the laws concerning rational and irrational exponents
- Explain the relation between exponents and logarithms
- Explain, prove and apply the laws of logarithms
- Explain the concept of exponential, logarithmic and absolute value functions and solve mathematical problem
- Sketch the graph of functions
- Represent exponential, logarithmic and absolute value Function by graphs
- Find logarithms and antilogarithms using calculators.

9·1 Integral, Rational and Irrational Exponents

We recall some notations :

N denotes the set of natural numbers

Z denotes the set of integers

Q denotes the set of rational numbers

R denotes the set of real numbers

If a is any real number and n is any natural number, then a is the product of n factors each equal to a. In symbols $a^n = a \cdot a \cdot a$............ (n factors)

In particular, $a^n = a$, $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$ and so on.

In the symbol $a^n . a$ is called the base and n is called the exponent or index. For example, In 3^4 the base is 3 and the exponent (index) is 4.

In $\left(\frac{2}{3}\right)^4$ base is $\frac{2}{3}$ and exponent is 4.

Definition : For all a $\in$ R

(1) $a^1 = a$

(2) $a^n = a.a.a..........a$ $\{n$ times factor of a) $n \in N, n > 1$

Note carefully that in a^2 a is multiplied with itself once ; in a^3, a is multiplied with itself twice. So the number multiplications is one less than the index.

Irrational Exponent :

We fixed the value of $a^x (a > 0)$ so that for any p the approximate value of x. a^p for a^x. For example we consider the number $3^{\sqrt{5}}$. We know $\sqrt{5}$ is a irrational number and $\sqrt{5} = 2 \cdot 236067977.........$ (Using calculator, we get the value of $\sqrt{5}$, it indicate as the decimal expression is infinite) approximate value of $\sqrt{5}$.

$p_1 = 2 \cdot 23$ $\quad p_2 = 2 \cdot 236$ $\quad p_3 = 2 \cdot 2360$

$p_4 = 2 \cdot 236067$ $\quad p_5 = 2 \cdot 2360679$ $\quad p_6 = 2 \cdot 23606797$

Considering the approximate value of $3^{\sqrt{5}}$

$q_1 = 3^{2 \cdot 23} = 11 \cdot 5872505.................$

$q_2 = 3^{2 \cdot 236} = 11 \cdot 6638822.................$

$q_3 = 3^{2 \cdot 2360} = 11 \cdot 6646510.................$

$q_4 = 3^{2 \cdot 236067} = 11 \cdot 6647407.................$

$q_5 = 3^{2 \cdot 2360679} = 11 \cdot 6647523.................$

$q_6 = 3^{2 \cdot 23606797} = 11 \cdot 6647532.................$

are found. This value also get by using calculator, $3^{\sqrt{5}} = 11 \cdot 664 \times 533..........$

9·2 Laws of Exponents (Indices)

Law 1: For every $a \in R$ and $n \in N$, we have

$$a^1 = a$$
$$a^{n+1} = a^n \cdot a.$$

Proof : By definition, $a^1 = a$ and $n \in N = a^{n+1} = \overbrace{\underbrace{a \cdot a \cdot a......}_{n \text{ factors in brackets}}}^{n+1 \text{ factors}} \quad a \cdot a = a^n \cdot a$

Law 2 : For every $n \in R$ and $m, n \in N$, we have

$$a^m \cdot a^n = a^{m+n}$$

Proof : a^m is the product of m factors each equal to a : a^n is the product of n factors each equal to a. So $a^m . a^n$ is the product of $m + n$ factors each equal to a. Therefore, $a^m \cdot a^n = a^{m+n}$.......(1)

This is the functional law about exponents (indices).

Law 3. For every $a \in R$, $a \neq 0$ and unequal natural numbers m and n, we have

$$\frac{a^m}{a^n} = \frac{a^{m-n} \text{ when } m > n}{\frac{1}{a^{n-m}} \text{ when } m < n}$$

Proof : Then $a \neq 0$ because the product of two or more non zero real numbers is nonzero. So, $\frac{a^m}{a^n}$ is defined.

(1) Suppose $m > n$ then $m - n$ is a natural number.

$\therefore \quad a^{m-n} \cdot a^n = a^{(m-n)+n} = a^m$ [Law 2]

Dividing both sides by a^n, we get $a^{m-n} = \frac{a^m}{a^n}$.

(2) Suppose, $m < n$ then $n - m$ a natural number

$\therefore \quad a^{n-m} \cdot a^m = a^{(n-m)+m} = a^n$ [Law 2]

Dividing both side by a^{n-m} and by a^n, we get

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$$

Law 4 : For every $a \in R$ and $m, n \in N$, we have $(a^m)^n = a^{mn}$

Proof : $(ab)^n$ is the product of n factors each equal to ab which is the product of a and b. We rearrange these factors as follows : we put all factors equal to a side by side ; then all factors equal to b side by side. So $(ab)^n$ is the product of n factors each equal to b. The first product to a^n, the second product is b^n. Therefore $(ab)^n$ = $a^n . b^n$ is proved.

Corollary : For any $a_1, a_2, \ldots, a_m$ and every $n \in R$ we have

$$(a_1 \cdot a_2 \ldots\ldots a_m)^n = a_1^n \cdot a_2^n \ldots\ldots a_m^n$$

Law 5 : $a \in R$ and $m, n \in N$ we have

$$(a \cdot b)^n = a^n \cdot b^n$$

Proof : taken $a_1 = a_2 = \ldots\ldots = a_m = a$ in Corollary above. Then

L.H.S. $(a^m)^n$ = and H.H.S. = $a^n \cdot a^n \ldots\ldots a^n$ [m factors]

$= a^{n+n+\ldots\ldots+n}$ [m summands in index]

$= a^{mn}$

We are ready to extend the definition of a^n to the case n = 0 and to negative integers. In all such cases a must not be zero.

Definition : For every $a \neq 0$ in R.

$$a^0 = 1, \; a^{-n} = \frac{1}{a^n} \text{ where } n \in N$$

These definitions are dictated by the requirement that the functional index law $a^m \cdot a^n = a^{m+n}$ would continue to hold for zero and negative values of m and n.

So this requires that $a^{-n} \cdot a^n$ be equal to $a^{0+n} = a^n$. This requirement is satisfied if and only if we define a^0 to be 1.

Similarly, $a^{-n} \cdot a^{n}$ should be equal to $a^{-n+n} = a^{0}$ =1. These requires that a^{-n} be equal to $\frac{1}{a^{n}}$, the reciprocal (multiplicative inverse) of a^{n}.

Example 1. $2^5 \cdot 2^6 = 2^{5+6} = 2^{11}$

$$\frac{3^5}{3^3} = 3^{5-3} = 2^2$$

$$\frac{3^3}{3^5} = \frac{1}{2^{5-3}} = \frac{1}{2^2}$$

$$\left(\frac{5}{4}\right)^3 = \frac{5}{4} \times \frac{5}{4} \times \frac{5}{4} = \frac{5 \times 5 \times 5}{4 \times 4 \times 4} = \frac{5^3}{4^3}$$

$$(4^2)^7 = 4^{2\times7} = 4^{14}$$

$$(a^2b^3)^5 = (a^2)^5 \cdot (b^3)^5 = a^{2\times5} \cdot b^{3\times5} = a^{10}b^{15}$$

Example 2. $6^o = 1, (-6)^o = 1, 7^{-1} = \frac{1}{7}.$

$$7^{-2} = \frac{1}{7^2} = \frac{1}{49}, 10^{-1} = \frac{1}{10} = 0 \cdot 1$$

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100}$$

Example 3. Assuming the validity of the law $(a^m)^n = a^{mn}$ for $m, n \in N$ Prove that $(a^m)^n = a^{mn}$ holds for all $m \in N$ and $n \in Z$ where $a \neq 0$.

Solution : We have $(a^m)^n = a^{mn}$ for all $m, n \in N$. Let $a \neq 0$.

Suppose $a \neq 0$. Then $a^m \neq 0$ so $(a^m)^o = 1$ and $a^{m^n} = a^o = 1$

So the claim holds for $n = 0$.

Now suppose n is a negative integer ; say $n = -k,$ where $k \in N$. Then

$$(a^m)^n = (a^m)^{-k} = \frac{1}{(a^m)^k} \quad \text{by definition of negative integral power.}$$

$$= \frac{1}{a^{mk}} = a^{-mk} = a^{m(-k)} = a^{mn}.$$

So the claim holds for all $m \in N$ and $n \in Z$.

Example 4. For all $m, n \in N$ and $\frac{a^m}{a^n} = a^{m-n}$ where $a \neq 0$

Solution : The claim is true in case $m > n$, $\frac{a^m}{a^n} = a^{m-n}$ [Law 3]

Suppose $m < n$. Then $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ [by definition of negative integral power]

$\therefore \quad \frac{a^m}{a^n} = a^{-(n-m)}$ [Law 4]

$= a^{m-n}$

Suppose $m = n$, then $\frac{a^m}{a^n} = \frac{a^n}{a^n} = 1 = a^o$ [Law 3]

$= a^{m-m} = a^{m-n}$

As a particular case of example 3 we not $(a^{-1}) = a^{-n} = (a^n)^{-1}$ for all n $\in$ Z where $n \neq 0$.

Example 5. Suppose $a \neq 0$; prove that the following index laws holds for all $m, n \in Z$

(i) $a^m \cdot a^n = a^{m+n}$

(ii) $\frac{a^m}{a^n} = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

(iv) $(ab)^n = a^n b^n$

(v) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

Solution : (i) We have proved that $a^m \cdot a^n = a^{m+n}$ holds when $m, n \in Z$ and when $m \in N$ and $n \in Z$. So it remains to prove it when m and n both are negative integers. Let m = -r, n = -s, $s \in Z$. Then

$$a^m.a^n.a^{-r}.a^{-s} = \frac{1}{a^r} \cdot \frac{1}{a^s} = \frac{1}{a^r.a^s} = \frac{1}{a^{r+s}}$$

$$= a^{-(r+s)} = a^{-r-s} = a^{m+n}.$$

(ii) We know $\frac{a^m}{a^n} = a^{m-n}$ holds for all $m, n \in N$.

The claim is true when m or n is 0, by definition of negative integral power.

Suppose $m < 0$ and $n >$ ay $m = -r$. Then

$$\frac{a^m}{a^n} = \frac{a^{-r}}{a^n} = a^{-r-n} \text{ by the already proven case}$$

$$= a^{m-n}$$

Suppose m > 0 and n < 0 ; say m = -s. Then

$$\frac{a^m}{a^n} = \frac{a^m}{a^{-s}} = a^{m-(-s)} \text{ by the already proven case}$$

$$= a^{m+n} = a^{m-n}.$$

Finally, suppose m < 0 and n < 0 ; say m = -r and m = -s. Then

$$\frac{a^m}{a^n} = \frac{a^{-r}}{a^{-s}} = n^{-r} \div a^{-s} = \frac{1}{a^r} \div \frac{1}{a^s}$$

$= \frac{1}{a^r} \cdot a^s = \frac{a^s}{a^r} = a^{s-r} = a^{-r-(-s)} = a^{m-n}$.

(iii) We know $(a^m)^n = a^{mn}$ holds when m > 0 and n > 0, also when $m \in N$, n is 0 or a negative integer. It remains to prove the claim when m < 0 and n < 0 ; say m = -r and n = -s. Then

L.H.S. = $(a^{-r})^{-s} = \frac{1}{(a^{-r})^s} = \frac{1}{a^{-rs}} a^{s-r}$

$= a^{rs} = a^{(-r)(-s)} = a^{mn}$ proved.

Activity :

1. Use mathematical induction to prove the index law $(a^m)^n = a^{mn}$ for all $n(n \in N)$
2. Use mathematical induction to prove the laws $(a \cdot b)^n = a^n . b^n$ for all $n \in N$.
3. Use mathematical induction to prove the laws $\left(\frac{1}{a}\right)^n = \frac{1}{a^n}$, where $a > 0$ and $n \in N$

   using the law $(a \cdot b)^n = a^n . b^n$ show that $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ where $a, b \in R$, $b > 0$, and $n \in N$
4. Suppose $a \neq 0$, and $m, n \in Z$, for any positive exponent justify $a^m \cdot a^n = a^{m+n}$ and show that $a^m \cdot a^n = a^{m+n}$ when (i) $m > 0$ and $n < 0$, (ii) $m < 0$ and $n < 0$

9·3 Rational Exponents

Before extending the definition of *n* to fractional exponents, we need to discuss the meaning of n-the root, where $n \in N$.

Definition : Given $a \in R$ and a natural number $n > 1$, a real number *x* is called n-th root of a if $x^n = a$. For n = 2 we call it square root, for n = 3 we call it cube root.

Example 5. (i) 2, as well as -2, is a fourth root of 16, because $(2)^4 = 16$ and $(-2)^4 = 16$

(ii) 3 is cube root of 27, while -3 is cube root of -27, because $(-3)^3 = -27$

(iii) For any n > 1, 0 is n-th root of 0, because $0^n = 0$.

(iv) –4 has no square root, because square of any real number is non-negative.

It is important to know and remember the following facts :

(1) Every positive real number has a unique n-th root, whether n is even or odd. This unique positive n-th root is denoted by $\sqrt[n]{a}$ and called the n-th root of *a*. For n = 2, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$.

(2) If $a < 0$ and n is odd, then a has *a* unique n-th root, which is negative ; the notation $\sqrt[n]{a}$ is used for this root.

(3) If n is even and a is negative, then a has no n-th root.

Note : So, $\sqrt{4} = 2$ is the positive square root of 4. For the negative square root we must write $\sqrt{4}$. Therefore, if *a* is any real number, then

$\sqrt{a^2} = a$ in case a > 0, and $\sqrt{a^2} = -a$, in case a is negative.

Recall that the absolute value of a is defined by $= |a| = \begin{cases} a, \text{ when } a > 0 \\ -a, \text{ when } a < 0 \end{cases}$

Therefore, $\sqrt{a^2} = |a|$

(1) If $a > 0$, $\sqrt[n]{a} > 0$

$a < 0$ and n is odd

$\sqrt[n]{a} = -\sqrt[n]{|a|} < 0$ [where $|a|$ is absolute value of a]

Example 6. $\sqrt[3]{-8} = 2$ and $\sqrt[3]{-8} = -2$; therefore $\sqrt[3]{-8} = -\sqrt[3]{-8}$

This is a special case of the following law.

Low 7. $a < 0$ and $n > 1$ is odd, then $\sqrt[n]{a} = -\sqrt[n]{|a|}$

Proof : Then $-a > 0$; so $-a$ has a unique n-th root, say x. $x^n = -a$

or, $-x^n = a$ $(-x)^n$ because n is odd. So $-x$ is the unique n-th root of $-a$; this proves the claim.

Example 7. $-\sqrt[3]{27}$

Solution : $-\sqrt[3]{27} = -\sqrt[3]{(3)^3} = -3$

Law 8 : For any $a > 0, n > 1$ and $m \in Z$, we have $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

Proof : Then, $\sqrt[n]{a} = x$ and $\sqrt[n]{a^m} = y$

$\therefore$ $x^n = a$ and $y^n = a^m$

$\therefore$ $y^n = a^m = (x^n)^m = (x^m)^n$

Where $y > 0, x^m > 0,$ principal value of n

Considering root, we get $y = x^m$

or, $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

or, $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

Law 9 : Let $a > 0$ and $\dfrac{m}{n} = \dfrac{p}{q}$, where $m, p \in Z$ and $n, q \in N, n > 1, q > 1$

Then, $\sqrt[n]{a^m} = \sqrt[q]{a^p}$

Proof : If $qm = pn$.

Let, $\sqrt[n]{a^m} = x \Rightarrow, x^n = a^m$

$\therefore$ $(x^n)^q = (a^m)^q$

$\therefore$ $x^{nq} = a^{mq} = a^{a\sqrt{m}} = a^{pn}$

or, $(x^q)^n = (a^p)^n$

$\therefore$ $x^q = a^p$ [by uniqueness of n-th root]

$\therefore$ $x = \sqrt[q]{a^p}$

$\therefore \quad \sqrt[n]{a^m} = \sqrt[q]{a^p}$

Remark : This result is vital : it shows that $\sqrt[n]{a^m}$ does not change if the fraction $\frac{m}{n}$ is replaced by an equivalent (that is, equal) fraction.

Corollary : For every $a > 0$ and $n, k \in N,\ n > 1$

We have, $\sqrt[n]{a} = \sqrt[nk]{a^k}$

We are now ready to define fractional exponents. First we define $a^{\frac{1}{a}}$ were $n \in$N and $a \in$R in such that $\sqrt[n]{a}$ exists (so a has to be > 0 in case n is even).

Definition : $a^{\frac{1}{a}} = \sqrt[n]{a}$ for every $n > 1$ and $a \in R$ for which $\sqrt[n]{a}$ exists.

Remark 1 : This definition is dictated by the requirement that the index law $(a^m)^n = a^{mn}$ should hold for fractional values of m and n. Then $\left(a^{\frac{1}{a}}\right)^n$ has to be = $\left(a^{\frac{1}{a}}\right)^n = a^1 = a$. So $a^{\frac{1}{a}}$ has to be the n-the $\sqrt[n]{a}$ is rational if and only if a is the n-th power of a rational number. Except in this case, we have to use an approximate value of $\sqrt[n]{a}$.

Definition : Suppose $n \in N$, $n > 1$ and $a \in R$ is such that $\sqrt[n]{a}$ exists. Then $a^{\frac{m}{n}}$ = $\left(a^{\frac{1}{a}}\right)^m$ for every m $\in$ Z.

Thus, $a^{\frac{m}{n}} = \sqrt[n]{a}^m = \left(a^{\frac{1}{a}}\right)^m$.

We have already pointed out that $a^{\frac{m}{n}}$ will not change if $\frac{m}{n}$ is replaced by an equal.

We now show that the index laws continue to hold for rational values of the exponents.

9·4 Rational Fractional exponent

Definition : If $a \in R$ and $n \in N,\ n > 1$ when (5) $a^{\frac{1}{n}} = \sqrt[n]{a}$ when $a > 0$ or $a < 0$ and odd.

Remark 1 : Laws of exponent $(a^m)^n = a^{mn}$

Is always true then $\left(a^{\frac{1}{n}}\right)^n = a^{\frac{n}{n}} = a^1 = a$ i.e., $a^{\frac{1}{n}}$ is the *n*the root. To avoid ambiguity we use the above mentioned definition.

Remark 2 : If $a < 0$ and $n \in N,\ n > 1$ and odd then from law we see that,

$a^{\frac{1}{n}} = \sqrt[n]{a} = -\sqrt[n]{|a|} = -|a|^{\frac{1}{n}}$

In this case we can determine the value of $a^{\frac{1}{n}}$ by this law.

Remark 3 : Though a is rational some cases $a^{\frac{1}{n}}$ is irrational. In this case we use approximate value of $a^{\frac{1}{n}}$.

Definition : If $a > 0, m \in Z$ and $n \in N,\ n > 1$, (6) $a^{\frac{m}{n}} = a^{\left(\frac{1}{n}\right)^m}$

Remark : Definition (5) and (6) and law 8 we see that, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$
where, $a > 0,\ m \in Z$ and $n \in N, n > 1$

Therefore $p \in Z$ and $q \in Z, n > 1$ if, $\frac{m}{n} = \frac{p}{q}$ from law 9 , we see that $a^{\frac{m}{n}} = a^{\frac{p}{q}}$

Remark 2 : we find the explanation of a^r from the definition of rational fractional exponent where, $a > 0$ and $r \in Q$. From the above discussion, if $a > 0$ and r is divided into equal fractional value a^r remain the same.

Remark 3 : the rules of law 6 is generally true for all exponent

Law 10. If $a > 0, b > 0$ and $r, s \in Q$ so

(a) $a^r \cdot a^s = a^{r+s}$ (b) $\frac{a^r}{a^s} = a^{r-s}$

(c) $(a^r)^s = a^{rs}$ (d) $(ab)^r = a^r b^r$

(e) $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$

Repeated application of (a) and (d) we see that,

Corollary : If (1) $a > 0$ and $r_1, r_2 .... r_k \in Q$ so,

$a^{r_1} \cdot a^{r_2} \cdot a^{r_3} .... a^{r_k} = a^{r_1+r_2+r_3+...........+r_k}$

(2) If $a_1 > 0, a_2 > 0 ....., a_n > 0$ and $r \in Q$ so, $(a_1 \cdot a_2 .... a_n)^r = a_1^r \cdot a_2^r .... a_n^r$.

Example 7. Show that, $a^{\frac{m}{n}} \cdot a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}}$
where, $a > 0; m, p \in Z; n, q \in N, n > 1, q > 1$.

Solution : $\frac{m}{n}$ and $\frac{p}{q}$

$a^{\frac{m}{n}} \cdot a^{\frac{p}{q}} = a^{\frac{mq}{nq}} \cdot a^{\frac{np}{nq}} = \left(a^{\frac{1}{nq}}\right)^{mq} \left(a^{\frac{1}{nq}}\right)^{np}$ [Definition 6]

$= \left(a^{\frac{1}{nq}}\right)^{mq+np}$ [law 6]

$= a^{\frac{mq+np}{nq}}$ [Definition 6]

$= a^{\frac{mq}{nq}+\frac{np}{nq}}$

$= a^{\frac{m}{n}+\frac{p}{q}}$

The following facts should be carefully noted :

(i) If $a^x = 1$ where $a > 0$ and $a \neq 1$ so $x = 0$

(ii) If $a^x = 1$ where $a > 0$ and $x \neq 0$ so $a = 1$

(iii) If $a^x = a^y$ where $a > 0$ and $a \neq 1$ so $x = y$

(iv) If $a^x = b^x$ where $\frac{a}{b} > 0$ and $x \neq 0$ so $a = b$

Example 8. Simplify :

If $a^x = b, b^y = c$ and $c^z = a$ show that, $xyz = 1$.

Solution : Given condition, $b = a^x$, $c = b^y$ and $a = c^z$

Now, $b = a^x = (c^z)^x = c^{zx} = (b^y)^{zx} = b^{xyz}$

$\Rightarrow b = b^{xyz} \Rightarrow b^1 = b^{xyz}$

$\therefore\ xyz = 1$. (Proved)

Example 9. If $a^b = b^a$ show that, $\left(\frac{a}{b}\right)^{\frac{a}{b}} = a^{\frac{a}{b}-1}$; and also prove that of $a = 2b$ if $b = 2$.

Solution : Given $a^b = b^a$

$\therefore\ b = (a^b)^{\frac{1}{a}} = a^{\frac{b}{a}}$

L.H.S. = $\left(\frac{a}{b}\right)^{\frac{a}{b}} = \left(\frac{a}{a^{\frac{b}{a}}}\right)^{\frac{a}{b}} = \left(a^1 \cdot a^{-\frac{b}{a}}\right)^{\frac{a}{b}}$

$= a^{\frac{a}{b}} \cdot a^{-1} = a^{\frac{a}{b}-1}$ R.H.S. (proved)

Again of $a = 2b$

$$\left(\frac{2b}{b}\right)^{\frac{2b}{b}} = (2b)^{\frac{2b}{b}-1} \Rightarrow (2)^2 = (2b)^{2-1}$$

$$\Rightarrow 4 = 2b \qquad \therefore \; b = 2 \text{ (Proved)}$$

Example 10. If $x^{x\sqrt{x}} = \left(x\sqrt{x}\right)^x$, find the value of x

Solution : Given $x^{x\sqrt{x}} = \left(x\sqrt{x}\right)^x = (x^x)^{\sqrt{x}} = \left(x \cdot x^{\frac{1}{2}}\right)^x = \left(x^{1+\frac{1}{2}}\right)^x$;

$$= \left(x^{\frac{3}{2}}\right)^x = \left(x^x\right)^{\frac{3}{2}}$$

$$\therefore \; \left(x^x\right)^{\sqrt{x}} = \left(x^x\right)^{\frac{3}{2}}$$

$$\Rightarrow \; \sqrt{x} = \frac{3}{2} \qquad \therefore \Rightarrow \; x = \left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

Example 11. If $a^x = b^y = c^z$ and $b^2 = ac$ show that $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}$

Solution : then $a^x = b^y$

or, $a = b^{\frac{y}{x}}$

Again, $c^z = b^y$ $\qquad \therefore \; c = b^{\frac{y}{z}}$

Now $b^2 = ac$

$$\therefore \; b^2 = b^{\frac{y}{x}} \cdot b^{\frac{y}{z}} = b^{\frac{y}{x}+\frac{y}{z}}$$

$$\Rightarrow \; 2 = \frac{y}{x} + \frac{y}{z} \Rightarrow \frac{1}{x} + \frac{1}{z} = \frac{2}{y} \text{ (proved).}$$

Example 12. Prove that, $\left(\frac{a^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a} \times \left(\frac{x^a}{x^b}\right)^{a+b} = 1$

Solution : L.H.S. = $\left(\frac{a^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a} \times \left(\frac{x^a}{x^b}\right)^{a+b}$

$= (x^{b-c})^{b+c} \times (x^{c-a})^{c+a} \times (x^{a-b})^{a+b}$

$= x^{b^2-c^2} \times x^{c^2-a^2} \times x^{a^2-b^2}$

$= x^{b^2-c^2+c^2-a^2+a^2-b^2}$

$= x^0$

= 1 = R.S.H. (proved)

Example 13. If $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$ and $abc = 1$ show that $x + y + z = 0$

Solution : Let $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}} = k.$

Than $a = k^x, b = k^y, c = k^z$

So, $abc = k^x k^y k^z = k^{x+y+z}$

Given $abc = 1$

$\therefore\ k^{x+y+z} = k^0$

$\therefore\ x + y + z = 0$

Example 14. Simplify : $\dfrac{1}{1+a^{y-z}+a^{y-x}} + \dfrac{1}{1+z^{z-x}+a^{z-y}} + \dfrac{1}{1+a^{x-y}+a^{x-z}}$

Here, $\dfrac{1}{1+a^{y-z}+a^{y-x}} = \dfrac{a^{-y}}{a^{-y}(1+a^{y-z}+a^{y-x})} = \dfrac{a^{-y}}{a^{-y}+a^{-z}+a^{-x}}$

Similarly, $\dfrac{1}{1+z^{z-x}+a^{z-y}} = \dfrac{a^{-z}}{a^{-z}(1+a^{z-x}+a^{z-y})} = \dfrac{a^{-z}}{a^{-z}+a^{-x}+a^{-y}}$

and $\dfrac{1}{1+a^{x-y}+a^{x-z}} = \dfrac{a^{-x}}{a^{-x}+a^{-y}+a^{-z}}$

Given quantity $\dfrac{1}{1+a^{y-z}+a^{y-x}} + \dfrac{1}{1+z^{z-x}+a^{z-y}} + \dfrac{1}{1+a^{x-y}+a^{x-z}}$

$= \dfrac{a^{-y}}{a^{-y}+a^{-z}+a^{-x}} + \dfrac{a^{-z}}{a^{-z}+a^{-x}+a^{-y}} + \dfrac{a^{-x}}{a^{-x}+a^{-y}+a^{-z}}$

$= \dfrac{a^{-x}+a^{-y}+a^{-z}}{a^{-x}+a^{-y}+a^{-z}} = 1$

Example 15. If $a = 2 + 2^{\frac{2}{3}} + 2^{\frac{1}{3}}$ show that, $a^3 - 6a^2 + 6a - 2 = 0$.

Solution : Gigen, $a = 2 + 2^{\frac{2}{3}} + 2^{\frac{1}{3}}$

$\therefore\ a - 2 = 2^{\frac{2}{3}} + 2^{\frac{1}{3}}$

or, $(a-2)^3 = \left(2^{\frac{2}{3}} + 2^{\frac{1}{3}}\right)^3$

$= 2^3 + 2 + 3 \cdot 2^{\frac{2}{3}} \cdot 2^{\frac{1}{3}}\left(2^{\frac{2}{3}} + 2^{\frac{1}{3}}\right)$

$= 6 + 6(a-2)\left[\because 2^{\frac{2}{3}} + 2^{\frac{1}{3}} = a - 2\right]$

or, $a^3 - 3a^2 2 + 3a \cdot 2^2 - 2^3 = 6 + 6a - 12$

or, $a^3 - 6a^2 + 12a - 8 = 6 + 6a - 12$

or, $a^3 - 6a^2 + 6a - 2 = 0$

Example 16. Solve : $4^x - 3\cdot 2^{x+2} \cdot + 2^5 = 0$

Solution : $4^x - 3\cdot 2^{x+2} \cdot + 2^5 = 0$

$\Rightarrow (2^2)^x - 3\cdot 2^x \cdot 2^2 + 2^5 = 0$

$\Rightarrow (2^x)^2 - 12\cdot 2^x + 32 = 0$

$\Rightarrow y^2 - 12y + 32 = 0$ [suppose $2^x = y$]

$\Rightarrow y^2 - 4y - 8y + 32 = 0$

$\Rightarrow y(y-4) - 8(y-4) = 0$

$\Rightarrow (y-4)(y-8) = 0$

$\therefore\ y - 4 = 0$ or $y - 8 = 0$

$\Rightarrow 2^x - 4 = 0$ [$\because 2^x = 7$] $\Rightarrow 2^x - 8 = 0$ [$\because 2^x = y$]

$\Rightarrow 2^x = 4 = 2^2$ $\Rightarrow 2^x = 8 = 2^3$

$\therefore\ x = 2$ $\therefore\ x = 3$

$\therefore\ x = 2, 3$

Activity :

1. Find the value of :

(i) $\dfrac{5^{n+2} + 35\times 5^{n-1}}{4\times 5^n}$ (ii) $\dfrac{3^4\cdot 3^8}{3^{14}}$

2. Show that, $\left(\dfrac{p^a}{p^b}\right)^{a^2+ab+b^2} \times \left(\dfrac{p^b}{p^c}\right)^{b^2+bc+c^2} \times \left(\dfrac{p^c}{p^a}\right)^{c^2+ca+a^2} = 1$

3. If $a = xy^{p-1}, b = xy^{q-1}$ and $c = xy^{r-1}$ then show that $a^{q-r}b^{r-p}c^{p-q} = 1$

4. Solve : (i) $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$.

(ii) $9^{2x} = 3^{x+1}$

(iii) $2^{x+3} + 2^{x+1} = 320$

5. Simplify : (i) $\sqrt[12]{(a^8)\sqrt{(a^6)\sqrt{a^4}}}$.

(ii) $\left[1 - 1\{1 - (1-x^3)^{-1}\}^{-1}\right]^{-1}$.

6. If $\sqrt[x]{a} = \sqrt[y]{b} = \sqrt[z]{c}$ and $abc = 1$ prove that $x + y + z = 0$.

7. If $a^m \cdot a^n = (a^m)^n$ prove that, $m(n-2) + n(m-2) = 0$.

Exercise 9·1

1. Prove that, $\left(a^{\frac{m}{n}}\right)^p = a^{\frac{mp}{n}}$ where m, $p \in Z$ and $n \in N$.

2. Prove that, $\left(a^{\frac{1}{m}}\right)^{\frac{1}{n}} = a^{\frac{1}{mn}}$ where $m, n \in Z$

3. Prove that, $(ab)^{\frac{m}{n}} = a^{\frac{m}{n}} b^{\frac{m}{n}}$, where $m \in Z$, $n \in N$

4. Show that, (a) $\left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right) = a - b$ (b)

$\frac{a^3 + a^{-3} + 1}{a^{\frac{3}{2}} + a^{\frac{-3}{2}} + 1} = \left(a^{\frac{3}{2}} + a^{\frac{-3}{2}} - 1\right)$

5. Simplify :

(a) $\left\{\left(x^{\frac{1}{a}}\right)^{\frac{a^2-b^2}{a-b}}\right\}^{\frac{a}{a+b}}$ (b) $\frac{a^{\frac{3}{2}} + ab}{ab - b^3} - \frac{\sqrt{a}}{\sqrt{a} - b}$

(c) $\frac{\left(\frac{a+b}{b}\right)^{\frac{a}{a-b}} \times \left(\frac{a-b}{a}\right)^{\frac{a}{a-b}}}{\left(\frac{a+b}{b}\right)^{\frac{b}{a-b}} \times \left(\frac{a-b}{a}\right)^{\frac{b}{a-b}}}$

(d) $\frac{1}{1 + a^{-m}b^n + a^{-m}c^p} + \frac{1}{1 + b^{-n}c^p + b^{-n}a^m} + \frac{1}{1 + c^{-p}a^m + c^{-p}b^n}$

(e) $\sqrt[bc]{\frac{x^{\frac{b}{c}}}{x^{\frac{c}{b}}}} \times \sqrt[ca]{\frac{x^{\frac{c}{a}}}{x^{\frac{a}{c}}}} \times \sqrt[ab]{\frac{x^{\frac{a}{b}}}{x^{\frac{b}{a}}}}$ (f) $\frac{(a^2 - b^2)^a (a - b^{-1})^{b-a}}{(b^2 - a^{-2})^b (b + a^{-1})^{a-b}}$

6. Show that,

(a) If $x = a^{q+r}b^p, y =^{r+p} b^q, z = a^{p+q}b^r$ show that $x^{q-r} \cdot y^{r-p} \cdot z^{p-q} = 1$.

(b) If $a^p = b, b^q = c$ and $c^r = a$ show that $pqr = 1$.

(c) If $a^x = p, a^y = q$ and $a^2 = (p^y q^x)^z$ show that $xyz = 1$.

7. (a) If $x\sqrt[3]{a} + y\sqrt[3]{b} + z\sqrt[3]{c} = 0$ and $a^2 = bc$ show that, $ax^3 + by^3 + cz^3 = 3axyz$.

(b) If $x = (a+b)^{\frac{1}{3}} + (a-b)^{\frac{1}{3}}$ and $a^2 - b^2 = c^3$ show that, $x^3 - 3cx - 2a = 0$

(c) If $a = 2^{\frac{1}{3}} + 2^{-\frac{1}{3}}$ show that, $2a^3 - 6a = 5$

(d) If $a^2 + 2 = 3^{\frac{1}{3}} + 3^{-\frac{2}{3}}$ and $a \geq 0$ show that, $a^3 + 9a = 8$

(e) If $a^2 = b^3$ show that, $\left(\frac{a}{b}\right)^{\frac{3}{2}} + \left(\frac{b}{a}\right)^{\frac{2}{3}} = a^{\frac{1}{2}} + b^{-\frac{1}{3}}$

(f) If $b = 1 + 3^{\frac{2}{3}} + 3^{-\frac{1}{3}}$ show that, $b^3 - 3b^2 - 6b - 4 = 0$

(g) If $a+b+c=0$ show that, $\frac{1}{x^b+x^{-c}+1}+\frac{1}{x^c+x^{-a}+1}+\frac{1}{x^a+x^{-b}+1}=1$.

8. (a) If $a^x=b, b^y=c$ and $c^z=1$ what is value of $xyz=$?

(b) If $x^a=y^b=z^c$ and $xyz=1$ what is value of $ab+bc+ca=$?

(c) If $9^x+(27)^y$ find the value of $\frac{x}{y}$?

9. Solve

(a) $3^{2x+z}+27^{x+1}=36$

(b) $5^x+3^y=9$
$5^{x-1}+3^{y-1}=2$

(c) $4^{3y-2}=16^{x+y}$
$3^{x+2y}=9^{2x+1}$

(d) $2^{2x+1}\cdot 27^{3y+1}=8$
$2^{x+2}\cdot 2^{y+2}=16$

9·6 Logarithm

Logarithm originated from two Greek Word Logos and arithmas. Logos means discussion and arithmas means number. That is discussion about special number.

Definition : If a is positive but not equal to 1, then $x=\log_a b$ means $a^x=b$. x is called the logarithm of *b* to the base *a*.

Observe that, by definition $a^x=b\Rightarrow x=\log_a b$

For example $\log_2 64=6$ because $2^6=64$ and $\log_8 64=2$ because $8^2=64$

So, the same number has different logarithm with respect to different bases.

Conversly, if $x=\log_a b\Rightarrow a^x=b$.

In this case the number b is the anti logarithm of x with respect to base a and we write $b=anti\log_a x$

If $\log a=n$, so a is the antilogarithm of n, $\log a=n$ thate is $a=anti\log n$.

Note : Since x^2 is positive for every $x\in R$, no negative number has a logarithm with respect to any base. Since $a^o=1$, we have $\log_a 1=0$.

On the other hand, every real number x, positive or negative, is the logarithm of som positive number, viz. of $a^x=b$, say. We then is b say the antilogarithm of *a* to the base *a*, and write $b=\text{anti}\log_a x$.

If $a>0$ I $a>1$ and $b\neq 0$ logaritim of b with base a can be denoted by $\log_a b$.

Therefore $\log_a b=x$ if and only if $a^x=b$

(a) $\log_a{}^{(ax)}=x$ (b) $a^{\log_a b}=b$

Example 1. $anti\log\ 2\cdot 82679=753$

$anti\log(9\cdot 82672-10=0\cdot 671$

and $anti\log(6\cdot 74429-10=0\cdot 000555$

Example 2. (1) $4^2 = 16 \Rightarrow \log_4 16 = 2$

(3) $5^{-2} = \frac{1}{5^2} = \frac{1}{25} \Rightarrow \log_5\left(\frac{1}{25}\right) = -2$

(4) $10^3 = 1000 \Rightarrow \log_{10}(1000) = 3$

(5) $7^{\log_7 9}$ $[\therefore\ a^{\log_a b} = b]$

(6) $18 = \log_2 2^{18}$ $[\because \log_a a^x = x]$

9·7 Laws of logarithms :

1. $\log_a a = 1$ and $\log_a 1 = 0$
2. $\log_a (M \times N) = \log_a M + \log_a N$
3. $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$
4. $\log_a (M)^N = N \ \log_a M$
5. $\log_a M = \log_b M \times \log_a b$

Proof of these laws will be found in General Mathematics.

Example 2. $\log_2 5 + \log_2 7 + \log_2 3 = \log_2(5.7.3) = \log_2 105$

Example 3. $\log_3 20 - \log_3 5 = \log_3 \frac{20}{5} = \log_3 4$

Example 4. $\log_5 64 = \log_5 2^6 = 6\log_5 2$

Note : (*i*) If $x > 0,\ y > 0$ and $a \neq 1$ then $x = y$ or, $\log_a x = \log_a y$

(*ii*) If $a > 1$ and $x > 1$ then $\log_a x > 0$

(*iii*) If $0 < a < 1$ and $0 < x < 1$ then $\log_a x > 0$

(*iv*) If $a > 1$ and $0 < x < 1$ then $\log_a x < 0$

Example 5. Find the value of x when

(*i*) $\log_{\sqrt{8}} x = 3\frac{1}{2}$

(*ii*) If $\log_{10}[98 + \sqrt{x^2 - 12x + 36}] = 2$

Solution : (*i*) when $\log_{\sqrt{8}} x = 3\frac{1}{3} = \frac{10}{3}$

$\Rightarrow x = (\sqrt{8})^{\frac{10}{3}} = \left(\sqrt{2^3}\right)^{\frac{10}{3}}$

$\Rightarrow x = \left(2^{\frac{3}{2}}\right)^{\frac{10}{3}} = 2^{\frac{3}{2}\cdot\frac{20}{3}} = 2^5 = 32$

$\therefore\ x = 32$

(*ii*) when $\log_{10}[98 + \sqrt{x^2 - 12x + 36}] = 2$

$\Rightarrow 98+\sqrt{x^2-12x+36}=10^2=100$

$\Rightarrow \sqrt{x^2-12x+36}=2$

$\Rightarrow x^2-12x+36=4$

$\Rightarrow (x-4)(x-8)=0$

$\therefore\ x=4$ ev $x=8.$

Example 6. Show that $a^{\log_k b-\log_k c}\times b^{\log_k c-\log_k a}\times c^{\log_k a-\log_k b}=1.$

Solution : Let $P=a^{\log_k b-\log_k c}\times b^{\log_k c-\log_k a}\times c^{\log_k a-\log_k b}$

Then,

$\log_k p=(\log_k b-\log_k c)\log_k a+(\log_k c-\log_k a)\log_k b+(\log_k a-\log_k b)\log_k c.$

$\Rightarrow \log_k p=0$ [simplifying]

$\Rightarrow P=k^0=1$

Example 7. Show that $x^{\log_a y}=y^{\log_a x}$

Proof : Let $p=\log_a y, q=\log_a x$

So, $a^p=y, a^q=x$

$\therefore\ (a^p)^q=y^q\Rightarrow y^q=a^{pq}$

and $(a^q)^p=x^p\Rightarrow x^p=a^{pq}$

$\therefore\ x^p=y^q\Rightarrow x\log_a y=ylob_a x$

Example 8. Find that $\log_a p\times\log_p q\times\log_q r\times\log_r b=\log_a b$

Solution : L.H.S. $=\log_a p\times\log_p q\times\log_q r\times\log_r b$

$=(\log_p q\times\log_a p)\times(\log_r b\times\log_q r)$

$=\log_a q\times\log_q b=\log_q b$ = R.H.S.

Example 9. Show that, $\dfrac{1}{\log_a(abc)}+\dfrac{1}{\log_a d(abc)}+\dfrac{1}{\log_a c(abc)}=1$

Solution : Let $\log_a(abc)=x, \log_b(abc)=y, \log_c(abc)=z$

Then $a^x=abc, b^y=abc, c^z=abc$

$\therefore\ a=(abc)^{\frac{1}{x}}, b=(abc)^{\frac{1}{y}}, c=(abc)^{\frac{1}{z}}$

Now, $(abc)^1=abc=(abc)^{\frac{1}{x}}(abc)^{\frac{1}{y}}(abc)^{\frac{1}{z}}$

$=(abc)^{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$

$\therefore\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=1$

or, $\dfrac{1}{\log_a(abc)}+\dfrac{1}{\log_b(abc)}+\dfrac{1}{\log_c(abc)}=1$

Example 10. If $P=\log_a(bc), q=\log_b(ca), r=\log_c(ab)$, then show that

$$\frac{1}{p+1}+\frac{1}{q+1}+\frac{1}{r+1}=1.$$

Solution : $1+P=1+\log_a(bc)=\log_a a+\log_a(bc)=\log_a(abc)$

Similarly, $1+q=\log_b(abc),\ 1+r=\log_c(abc)$

Using the result of example (9) we get, $\dfrac{1}{\log_a(abc)}+\dfrac{1}{\log_b(abc)}+\dfrac{1}{\log_c(abc)}=1$

$\therefore\ \dfrac{1}{p+1}+\dfrac{1}{q+1}+\dfrac{1}{r+1}=1.$

Example 11. If $\dfrac{\log a}{y-z}+\dfrac{\log b}{z-x}+\dfrac{\log c}{x-y}=1$, then show that $a^x b^y c^z=1$

Solution : Let, $\dfrac{\log a}{y-z}+\dfrac{\log b}{z-x}+\dfrac{\log c}{x-y}=k$

Then, $\log a=k(y-z), \log b=k(z-x), \log c=k(x-y)$

$\therefore\ x\log a+y\log b+z\log c=k(xy-zx+yz-xy+zx-yz)=0$

or, $\log_a x+\log_a y+\log_a z=0$

or, $\log(a^x b^y c^z)=0$

or, $\log(a^x b^y c^z)=\log 1 \quad [\log 1=0]$

$\therefore\ a^x b^y c^z=1$ shown

Activity :

1. If $\dfrac{\log a}{b-c}=\dfrac{\log b}{c-a}=\dfrac{\log c}{a-b}$ then find the value of $a^a.b^b.c^c$.
2. If a, b, c are three consecutive integers, then prove that $\log(1+ac)=2\log b$
3. If $a^2+b^2=7ab$, the show that $\log\left(\dfrac{a+b}{3}\right)=\dfrac{1}{2}\log(ab)=\dfrac{1}{2}(\log a+\log b)$
4. If $\log\left(\dfrac{x+y}{3}\right)=\dfrac{1}{2}(\log x+\log y)$, then show that $\dfrac{x}{y}+\dfrac{y}{x}=7$
5. If $x=1+\log_a bc, y=1+\log_b ca$ and $z=1+\log_c ab$, then show that $xyz=xy+yz+zx$
6. (a) if $2\log_8 A=p,\ 2\log_2 2A=q$ and $q-p=4$, then find the value of A .
   (b) If $\log x^y=6$ and $\log 14x^{8y}=3$, then find the value of x .
7. Using table of logarithm (see General Mathematics), find the approximate value of P , where
   (a) $P=(0.087721)^4$
   (b) $P=\sqrt[3]{30\cdot 00618}$

9·7 Exponential, Logarithmic and Absolute Value Functions

Given a positive real number $a \neq 1$, the function $f(x) = a^x$ is called the exponential function to the base *a*.

The function $f(x) = \log_a x$ is called the logarithmic function to the base *a*. The function defined by $f(x) = (x)$ is called the absolute value function. We recall that $\{x\} = x$ if $x \geq 0$, $(x) = -x$ if $x < 0$.

In all numerical work, logarithm to the base 10 are used ; these are common logarithms. In all theoretical work, the base is an irrational number denoted by e. It is defined as the sum of the infinite series $e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + ...;$

Its value is approximately 2.71.

Graphs of Exponential, Logarithmic and Absolute Value Function.

Observe the following three tables of corresponding values of *x* and *y* :

Table 1 :

x	-2	-1	0	1	2	3
y	-4	-2	0	2	4	6

Table 2 :

x	0	1	2	3	4	5
y	1	3	9	27	81	243

Table 3 :

x	0	1	2	3	4	5	6	7	8	9	10
y	1	2	4	8	16	32	64	128	256	512	1024

In table 1. We notice that as x takes on the values – 2, – 1, ……, 2, 3, the corresponding values of y are such that the difference between two succeeding values is the same, viz. 2. This means that the six points (–2, 4), (–1, –2), ….., (2, 4), (3, 6), lie on a straight line. So Table 1 describes a linear function. In table 2 we notice that as takes on the values 0, 1, 2, 3, 4, 5, the corresponding values of y are the respective powers of 3 : 3^o, 3^1, 3^2, 3^3, 4^4, 5^5.

Thus, Table 2 lists the values of the exponential function $y = 3^x$.

Similarly, Table 3 lists the values of the exponential function $y = 2^x$.

> For exponentials function $f(x) = a^x$ is defined for all real number x when, $a > 0$ and $a \neq 1$

The functions, $y = 2^x, 10^x, x^x, e^x$ are exponential function

Activity :

Write the exponential function described in the following table.

1.	x	-2	-1	0	1	2
	y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

2.	x	-1	0	1	2	3
	y	-3	0	3	6	9

3.	x	1	2	3	4	5
	y	4	16	64	256	1024

4.	x	-3	-2	-1	0	1
	y	0	1	2	3	4

5.	x	-2	-1	0	1	2	6.	x	1	2	3	4	5
	y	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25		y	5	10	15	20	25

Which of the following functions are exponential functions ?

7. $y=-3^x$ 8. $y=3x$ 9. $y=-2x-3$ 10. $y=5-x$

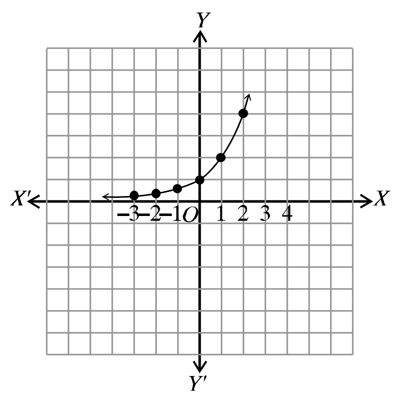
11. $y=x^2+1$ 12. $y=3x^2$

Graph of Function $f(x)=2^x$ **:**

To draw the graphs of given function, we list the values of x and y.

x	-3	-2	-1	0	1	2
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

Ploting the ordered pair in the graph we have the following graph



Here, domain $=(-\alpha,\alpha)$

Range $=(0,\alpha)$

From the figure we see that when $x=0$ then $y=2^{\circ}=1$ so the line passing through the point $(0,1)$.

Agin for any negative value of x, y is very near to 0, but not (0).

Similarly for any negative value of x, the value of y is ingcreasing,

Domain $(D)=(-\infty,\infty)$

Range $(R)=(0,\infty)$

Activity : Draw the graphs for $-3\le x\le 3$

1. $y=2^{-x}$ 2. $y=4^x$ 3. $y=2^{\frac{x}{2}}$ 4. $y=\left(\frac{3}{2}\right)^x$

As the exponent function is one one, therefore it has inverse.

$f(x)=\quad y=a^x$ exponential form

$f^{-1}(y)=\; x=a^y$ changing x and y

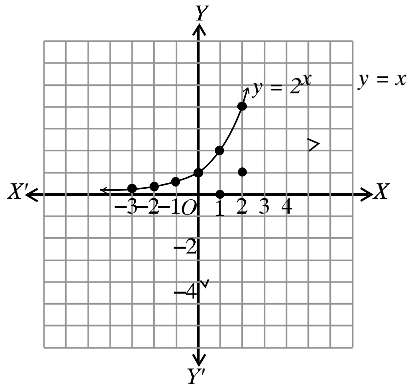
That is x is the logarithm of y with base a.

Definition : Logarithmic function $f(x)=\log_a x$ is defined where $a>0$ and $a\neq 1$

$f(x)=\log_3 x, Inx, \log_{10} x$ etc logarithmic function

Graphs of : $y=\log_2 x$

as $y = \log_2 x$ is the inverse function of $y = 2^x$.
Logarithmic function is determined as reflection of exponential function which is similar with respect to $y = x$.



Domain $(R) = (0, \infty)$
Range $(D) = (-\infty, \infty)$

Activity :
Draw the graphs and find its inverse function.

1. $y = 3x + 2$ 2. $y = x^2 + 3$ 3. $y = x^3 - 1$ 4. $y = \dfrac{4}{x}$
5. $y = 3x$ 6. $y = \dfrac{2x+1}{x-1}$ 7. $y = 2^{-x}$ 8. $y = 4^x$

Example 1. Find the domain and range of the function $f(x) = \dfrac{x}{|x|}$.

Solution : The formula of $f(x)$ entails division by $f(0) = \dfrac{0}{|0|} = \dfrac{0}{0}$ which is undefined.
$\therefore$ The given function does not exists at $x = 0$
The function is exists for all values of x except zero
Here domain, $D_f = R - \{0\}$

Again, $f(x) = \dfrac{x}{|x|} = \begin{cases} \dfrac{x}{x} \text{ when } x > 0 \\ \dfrac{x}{-x} \text{ when } x < 0 \end{cases}$
$= \dfrac{1 \text{ when } x > 0}{-1 \text{ when } x < 0}$

$\therefore$ Range, $R_f = \{-1, 1\}$

Example 2. Find the domain and range of the function $y = f(x) = \text{In}\, \dfrac{a+x}{a-x}, a > 0$.

Solution : Since any logarithm to base is defined only for positive numbers, we need to ascertain when $\dfrac{a+x}{a-x}$ is > 0.

Now, $\dfrac{a+x}{a-x} > 0$ is $\Leftrightarrow$ either (i) $a+x>0$ and $a-x>0$

or. (ii) $a+x<0$ and $a-x<0$.

Case (i) $\Rightarrow x > -a$ and $a > x$

$\Rightarrow -a < x$ and $x < a$

$\therefore$ Domain = $\{x : -a < x\} \cap \{x : x > a\}$

$= (-a, \infty) \cap (-\infty, a) = (-a, a)$

(ii) $\Rightarrow x < -a$ and $a < x$

$\Rightarrow x < -a$ and $x > a$

$\therefore$ Domain $\{x : x < -a\} \cap \{x : x > a\} = \phi.$

$\therefore$ Domain of the given function

$\therefore$ $D_f = from(i)$ and (ii) $(-a, a) \cup \phi = (-a, a)$

Range : $y = f(x) = In\dfrac{a+x}{a-x} \Rightarrow e^y = \dfrac{a+x}{a-x}$

$\Rightarrow a + x = ae^y - xe^y$

$\Rightarrow (1 + ae^y)x = a(xe^y - 1)$

$\Rightarrow x = \dfrac{a(e^y-1)}{e^y+1}$

x is real for all real values of y

$\therefore$ Range of the given function $Rf = R$

Activity : Find the domains and range of the following functions :

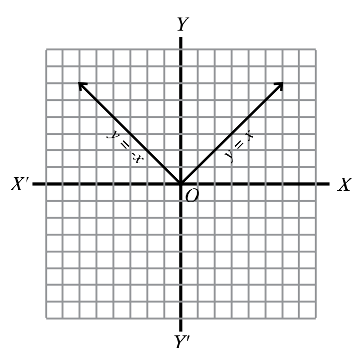
1. $y = In\dfrac{2+x}{2-x}$ 2. $y = In\dfrac{3+x}{3-x}$ 3. $y = In\dfrac{4+x}{4-x}$ 4. $y = In\dfrac{5+x}{5-x}$

Absolute Value :

In secondary mathematics absolute value is discussed elaborately, here definition only given.

For any real number x is zero, positive or negative. But absolute value of ‘ x is always zero or positive. Absolute value of x is expressed by $|x|$

$$|x| = \begin{cases} x \text{ when } x > 0 \\ -x \text{ when } x < 0 \end{cases}$$



For Example : $|0| = 0, |3| = 3, |-3| = -(-3) = 3$

Absosute Value Fuction

if $x \in R$ so

$$y = f(x) = |x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

Is called absolute value function

$\therefore$ Domain $= R$ and range $Rf = [0, \infty]$

Example 3. Find domain and range of the function

$f(x) = e^{\frac{|x|}{2}}$ when $-1 < x < 0$

Solution : $f(x) = e^{\frac{-|x|}{2}}, -1 < x < 0$

as x is fixed from -1 to 0

therefore domain, $D_f = (-1, 0)$

Again in the interval, $-1 < x < 0$, $f(x) \in \left(e^{\frac{-1}{2}}, 1\right)$

Therefore range, $f = \left(e^{-\frac{1}{2}}, 1\right)$

9·8 Graphs of function

Function is detected by representing geometrically in a plane. This geometrical representation is the drawing of graphs. Here we discussed the method of drawing graphs of exponential, logarithmic and absolute value function.

Draw the graphs of

(1) $y = f(x) = a^x$

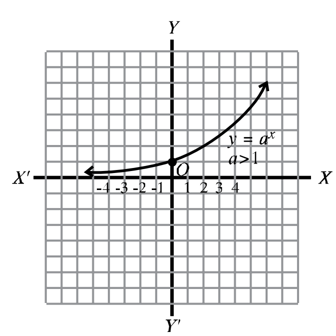
(i) When $a > 1$ and x is any real number then $f(x) = a^x$ is always positive.

Step 1 : For positive value of x, the value of $f(x)$ is increased with the increase of x

Step 2 : When $x = 0$, $y = a^{\circ} = 1$,

Therefore, $(0, 1)$ is a point on the line.

Step 3 : x is positive and x is monotone increasing and then y is monotone decreasing $x \to \infty$, $y \to 0$



Here $y = a^x, a > 1$ now the graph of the function is drawan in figure 1

Here $D_f = (-\infty, \infty)$ and $R_f = (-\infty, \infty)$

(ii) When x is real and $o < a < 1$, $y = f(x) = a^x$ is always positive

Step 1 : Observe if the value of x is increasing from the right side of origin, that is $x \to \infty$, $y = 0$

Step 2 : When $x = 0$, $y = a^{\circ} = 1$

Therefore the point $(0, 1)$ lies on the line

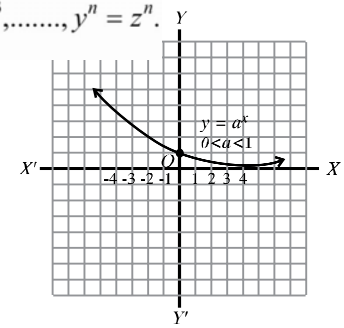
Step 3 : When $a < 1$, the value of x is negative and monotone increasing to the left side from the origin then y is monotone increasing , that is $y \to \infty$.

[Let

$a = \frac{1}{2} < 1, x = -2, -3, \ldots\ldots n,$ then $y = f(x) = a^x = \left(\frac{1}{2}\right)^{-2} = 2^2, y = 2^3, \ldots\ldots, y^n = z^n.$

If $n \to \infty$, $y \to \infty$]

Now the graph of $y = f(x) = a^x$, $0 < a < 1$ is drawn in fig 2



Here $D_f = (-\infty, \infty)$ and $R_f = (o, \infty)$

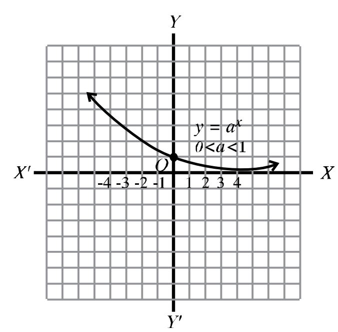
Activity : Sketch the graphs of the following functions ; mention their domains and ranges :

(i) $f(x) = 2^x$ (ii) $f(x) = \left(\frac{1}{2}\right)^x$ (iii) $f(x) = e^x, \quad 2 < e < 3.$

(iv) $f(x) = e^{-x}, \quad 2 < e < 3.$ (v) $f(x) = 3^x$

2. Draw the graphs of $f(x) = \log a^x$

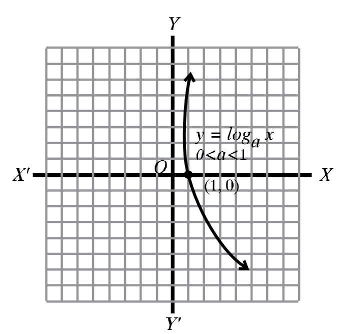
(i) Let, $y = f(x) = 10y_a x$ when $0 < a < 1$, the function can be written as $x = a^y$



a : y is strictly monotone decreasing ; this means, $y_2 < y_1$ whenever $x_2 < x_1$. y can be made as small as we place by choosing a large enough positive value of *x*. Symbolically, $y \to -\infty$ or, $x \to \infty$

b : As *x* takes on larger and larger negative values, *y* becomes larger and larger. *y* can be made as large as we place by choosing a large enough value of *x*. Symbolically, $y \to -\infty$ or, $x \to \infty$

Now from figure 3, $y = \log_a x$, $0 < a < 1$ is shown



(2) $y = \log_a x, a > 1.$

Here $D_f = (0, \infty)$ and $R_f = (-\infty, \infty)$

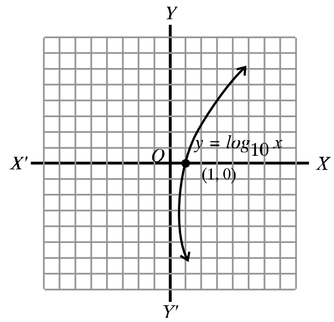
When $y = \log_a x, a > 1$

a : y becomes larger and larger negative as *x* takes on smaller and smaller positive values ; *y* can be made as large as we place by choosing a small positive value of *x*. Symbolically, $y \to -\infty$ or, $x \to \infty$

b : y becomes larger and larger negative as becomes larger and large positive. *y* can be made as large as negative number as we place by choosing a large enough value of *x*. Symbolically, $y \to -\infty$ or, $x \to \infty$

Now the graphs of $f(x) = \log a^n, a > 1$ is shown

Here $Df = (0, \infty)$ and $Rf = (-\infty, \infty)$



Example 3. Show the graph of $f(x) = \log_{10} x$.

Solution : $10^o = 1$; so, $y = f(x) = \log_{10} 1 = 0$. Therefore the graph passes through the point (1, 0). $y = 1$ when $x = 10$, $y = -1$ when $x = 0.1$.

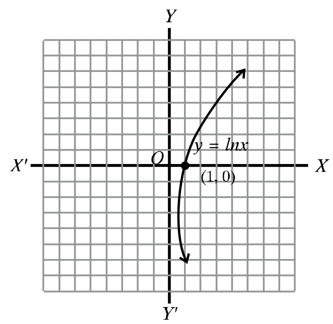
$y \to \infty$ as $x \to \infty$; and $y \to \infty$ as $x - 0^+$.

$D_f = (0, \infty)$ and $R_f = (-\infty, \infty)$

Example 4. Draw the graph of *f(x)* = In *x*.

Solution : Let $y = f(x) = Inx$. In 1 = 0 ; the graph passes through the point (1, 0).

$y \to \infty$ as $x \to \infty$ and $y \to \infty$ as $x \to 0^+$. The graph is shown below.



Activity :

1. The following table gives values of $y = \log_{10} x$ for certain values of *x*. Draw the graph $y = \log_{10} x$ from these data.

x	.5	1	2	3	4	5	10	12
y	–.3	0	0.3	0.5	.0	.7	1	1.0

2. Draw up a table of values of *y* = In *x* using a calculator ; hence draw the graph of *y* = In *x*.

Exercise 9·2

1. Which is the simplest form of the expression $\left\{\left(x^{\frac{1}{a}}\right)^{\frac{a^2-b^2}{a-b}}\right\}^{\frac{a}{a-b}}$?

   (a) 0 (b) 1 (c) a (d) x

2. If $a, b, p > 0$ and $a \neq 1, b \neq 1$ then
   i. $\log_a P = \log_b P \times \log_a b$
   ii. $\log_a \sqrt{a} \times \log_b \sqrt{b} \times \log_c \sqrt{c}$ is 2
   iii. $x^{\log_a y} = y^{\log_a x}$
   Which combination of these statements is correct ?
   (a) i and ii (b) ii and iii (c) i and iii (d) i, ii and iii
   Given that $a^x = b^y = c^z$, where $x, y, z \neq 0$ answer questions 3, 4, 5.
3. Which is correct ?
   (a) $a = b^{\frac{y}{z}}$ (b) $c^{\frac{z}{y}}$ (c) $a = c^{\frac{z}{x}}$ (d) $a \neq \frac{b^2}{c}$
4. Which of the following is equal to ac.
   (a) $b^{\frac{y}{x}} \cdot b^{\frac{y}{z}}$ (b) $b^{\frac{y}{x}} \cdot b^{\frac{z}{y}}$ (c) $b^{\frac{y}{x}+\frac{z}{y}}$ (d) $b^{\frac{z}{y}+\frac{y}{z}}$
5. If $b^2 = ac$, which one is correct ?
   (a) $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}$ (b) $\frac{1}{x} + \frac{1}{y} = \frac{2}{z}$ (c) $\frac{1}{y} + \frac{1}{z} = \frac{2}{x}$ (d) $\frac{1}{x} + \frac{1}{y} = \frac{2}{z}$
6. Show that,
   (a) $\log_k\left(\frac{a^n}{b^n}\right) + \log_k\left(\frac{b^n}{c^n}\right) + \log_k\left(\frac{c^n}{a^n}\right) = 0$
   (b) $\log_k(ab)\log_k\left(\frac{a}{b}\right) + \log_k(bc)\log_k\left(\frac{b}{c}\right) + \log_k(ca)\log_k\left(\frac{c}{a}\right) = 0$
   (c) $\log_{\sqrt{a}} b \times \log_{\sqrt{b}} c \times \log_{\sqrt{c}} a = 8$
   (d) $\log_a \log_a \log_a \left(a^{a^{a^b}}\right) = b$
7. (a) hw` $\frac{\log_k a}{b-c} = \frac{\log_k b}{d-a} = \frac{\log_k c}{a-b}$, show that $a^a b^b c^c = 1$
   (b) If $\frac{\log_k a}{y-z} = \frac{\log_k b}{z-x} = \frac{\log_k c}{x-y}$, show that
   (1) $a^{y+z} b^{z+x} c^{x+y} = 1$
   (2) $a^{y^2+yz+z^2} \cdot b^{z^2+zx+x^2} \cdot c^{x^2+xy+y^2} = 1$.
   (c) If $\frac{\log_k(1+x)}{\log_k x} = 2$, show that $x = \frac{1+\sqrt{5}}{2}$

(d) show that, $\log_k = \dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 2\log_k\left(x-\sqrt{x^2-1}\right)$

(e) If $a^{3-x}b^{5x} = a^{5+x}b^{3x}$, show that $x\log_k\left(\dfrac{b}{a}\right) = \log_k a$

(f) If $xy^{a-1} = P, xy^{b-1} = q, xy^{c-1} = r$, show that

$$(b-c)\log_k p + (c-a)\log_k q + (a-b)\log_k r = 0$$

(g) If $\dfrac{ab\log_k(ab)}{a+b} = \dfrac{bc\log_k(bc)}{b+c} = \dfrac{ca\log_k(ca)}{c+a}$, show that $a^a = b^b = c^c$

(h) If $\dfrac{x(y+z-x)}{\log_k x} = \dfrac{y(z+x-y)}{\log_k y} = \dfrac{z(x+y-z)}{\log_k z}$, show that $x^y y^z = y^z z^y = z^x x^z$

8. Using table of logarithms (see General Mathematics), find an approximate value of P, where.

(a) $P = 2\pi\sqrt{\dfrac{l}{g}}$ where $\pi \approx 3\cdot1416, g = 981$ and $l = 25\cdot5$

(b) $P = 10000 \times e^{005t}$ where $e = 1\cdot718$ and $t = 13\cdot86$

9. Using the formula $\ln P \approx 2\cdot3026 \times \log p$, find an approximate value of P, where

(a) $P = 10000$ (b) $P = \cdot001e^2$ (c) $P = 10^{100} \times \sqrt{e}$

10. Draw the graphs of the following functions :

(a) $y = 3^x$ (b) $y = -3^x$ (c) $y = 3^{x+1}$ (d) $y = -3^{x+1}$ (e) $y = 3^{-x+1}$ (f) $y = 3^{x-1}$

11. Write down the inverse function in each case, mention the domain and range and draw the graph :

(a) $y = 1 - 2^{-x}$ (b) $y = \log_{10} x$ (c) $y = x^2$

12. Find the domain and range of the function $f(x) = In(x-2)$ . . . . . D_f and R_f.

13. Find the domain and range of the function $f(x) = In\dfrac{1-x}{1+x}$.

14. Draw the graph of each function, mentioning the domain and range :

(a) $f(x) = |x|$ when $-5 \le x \le 5$

(b) $f(x) = x + |x|$ when $-2 \le x \le 2$

(c) $f(x)\dfrac{|x|}{x}$ when $x \ne 0$

0 when $x = 0$

(d) $f(x) = \dfrac{x}{|x|}$

(e) $f(x) = \log \frac{5+x}{5-x}, -5 < x < 5$

15. Given $x = \log_a y$ and $a > 0,\ a \neq 1$

(a) Convert (i) and (ii) into linear equation in x and y.

(b) Solve the equations and verify correctness.

(c) Should x and y be lengths of two adjacent sides of a quadrilateral and the angle included them be 90^o, state whether the quadrilateral would be a rectangle or a square ; find its area and the lengths of the diagonals.

16. Given $\frac{\log(1+x)}{\log x} = 2$

(a) Convert the given equation into a quadratic equation in x.

(b) Solve the quadratic equation and prove without calculation, that the square of each root of the quadratic equation exceeds itself by 1.

(c) Prove without calculation, that the square of each root of the quadratic equation exceeds itself by 1. Identify which of its root satisfy the given equation.

17. Given $y = 2^x$

(a) Mention the domain and range of the function.

(b) Draw the graph of the function, and mention its salient features.

(c) State whether the given function has an inverse function. If so, is it one-one ? Draw the graph of the inverse function.

Chapter Ten

Binomial Expansion

In our previous classes addition, subtraction, multiplication, division, square and cube related algebraic expression (Single term, binomial and polynomial) have been dicussed. If the power of binomial or polynomial expression is more than three, in that case determining the value of such expression is labourious and time consuming. If the power is more than three in which method the work is completed that will be presented in this chapter. Generaly, formula will be demonstrated for the power of n. By which the value of binomial expression of non negative integer of power will be possible to determine. But at this stage the value of n will not exceed a definite limit $n \leq 8$. A triangle will be introduce named as pascal's triangle so that students can easily understand and use the subject matter. The power of binomial expression may be positive or negative integer or fraction. But our present discussion will be limited in only positive integer of power. In next classes detailed discussion will be included.

After completing this chapter, the students will be able to-

- Describe the binomial expression
- Describe the pascal's triangle
- Describe the binomial expression for general power
- Find the value of $n!$ and ^nC_r
- Solve mathematical problem using binomial expression.

10·1 Expansion of $(1+y)^n$

The algebraic expression consists of two term is called binomial expression.|
$a+b,\ x-y,\ 1+x,\ 1-x^2,\ a^2-b^2$ etc are binomial expression. We first consider a binomial expression of $(1+y)$. Now we multiply $(1+y)$ by $(1+y)$ successively then we get $(1+y)^2,\ (1+y)^3,\ (1+y)^4,\ (1+y)^5$........... etc.

We know,

$$(1+y)^2 = (1+y)(1+y) = 1+2y+y^2$$

$$(1+y)^3 = (1+y)(1+y)^2 = (1+y)(1+2y+y^2) = 1+3y+3y^2+y^3$$

Similarly it is possible to determine the value of $(1+y)^4, (1+y)^5$...... etc by the process of lengthy multiplication. But it will be lengthy and time consuming of the powers of $(1+y)$ is increasing. So it will be better to find out a method for determining any powers of $(1+y)$. We easily determine the expansion of $(1+y)^n$ for power n. For n = 0, 1, 2, 3, 4, ... i'e for non negative value of n, our discussion is limited in this context. Now we carefully observe the proccedure.

Value of n		**Pascal's triangle**	**Number of terms**
$n=0$	$(1+y)^{\circ} =$	1	1
$n=1$	$(1+y)^1 =$	$1+y$	2
$n=2$	$(1+y)^2 =$	$1+2y+y^2$	3
$n=3$	$(1+y)^3 =$	$1+3y+3y^2+y^3$	4
$n=4$	$(1+y)^4 =$	$1+4y+6y^2+4y^3+y^4$	5
$n=5$	$(1+y)^5 =$	$1+5y+10y^2+10y^3+5y^4+y^5$	6

On the basis of the above expression, we come to the decision of the expansion of $(1+y)^n$.

Decision :

(a) In the expansion of $(1+y)^n$ the number of term is $(n+1)$ i'e number of terms is greater than from the power.

(b) The power of y is increasing from 0 to $1, 2, 3, \ldots\ldots\ldots, n$ i'e power of y is increasing orderly up to n .

Binomial coefficient : The coefficient of different powers of y on the above expansion is called binomial coefficients. One is considered as the coefficient of y . If we arrange the coefficients of the above expansion.

$n=0$ 1

$n=1$ 1   1

$n=2$ 1   2   1

$n=3$ 1   3   3   1

$n=4$ 1   4   6   4   1

$n=5$ 1   5   10   10   5   1

If we observe, the coefficients formed a triangular shape. The technique of determining the coefficients of binomial expansion is used first by Blaise pascal. So it is called pascal's triangle. We can easily determine the coefficients of binomial expansion by pascal's triangle.

Use of pascal's triangle

From pascal's triangle, we see that `1' is in both left and right side. The middle term of the triangle is the summation of two number just above the numbers. If we observe the following example, we can easily understand it.

Binomial coefficient for $n=5$ is 1 5 10 10 5 1

For $n=6$

n=5   1   5   10   10   5   1

n=6   1   6   15   20   15   6   1

$\therefore (1+y)^5 \quad = 1 + 5y + 10y^2 + 10y^3 + 5y^4 + y^5$

$\therefore (1+y)^6 \quad = 1 + 6y + 15y^2 + 20y^3 + 15y^4 + 6y^5 + y^6$

And $(1+y)^7 = 1 + 7y + 21y^2 + 35y^3 + 35y^4 + 21y^5 + 7y^6 + y^7$

> **Activity :** Find the expansion of the following (Using the above expansion) :
>
> $(1+y)^8 =$
>
> $(1+y)^9 =$
>
> $(1+y)^{10} =$

If we carefully observe, we will understand that this method has a special weakness. For example if we know the expansion of $(1+y)^5$, we need to know the expansion of $(1+y)^4$. Again it is necessary to know just above preceeding two coefficients for any binomial coefficients. For getting relief from this position we directly find out the technique for determining the binomial coefficients. From pascal's triangle we see that power of coefficient of binomial expansion depends on power n and position of the term. We consider a new symbol $\binom{n}{r}$ where the power 'n' and the position of the term r are related. For example if $n = 4$, the number of terms will be 5. Suppose the five terms are T_1, T_2, T_3, T_4, T_5 respectively.

We write the terms in the following way

Where $n = 4$ number of terms is 5 : T_1, T_2, T_3, T_4, T_5 .

Coefficients of them are : 1 4 6 4 1

Using new symbol : $\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$

Here, $\binom{4}{0} = 1, \quad \binom{4}{1} = \frac{4}{1} = 4$

$$\binom{4}{2} = \frac{4\times 3}{1\times 2} = 6, \quad \binom{4}{3} = \frac{4\times 3\times 2}{1\times 2\times 3} = 4 \text{ and } \binom{4}{4} = \frac{4\times 3\times 2\times 1}{1\times 2\times 3\times 4} = 1$$

$n = 1$	$\binom{1}{0} \quad \binom{1}{1}$
$n = 2$	$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$
$n = 3$	$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$
$n = 4$	$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$
$n = 5$	$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$

Therefore from the triangle above we can easily say that coefficient of third term (T_{2+1}) of $(1+y)^4$ is $\binom{4}{2}$ and coefficient of third (T_{2+1}) and fourth (T_{3+1}) of $(1+y)^5$ are $\binom{5}{2}$ and $\binom{5}{3}$ respectively. Generally, coefficients of r th term T_{r+1} of $(1+y)^n$ is $T_{r+1} = \binom{n}{r}$.

Now, to know the value of $\binom{n}{r}$, we again observe the pascal's triangle. From the two sides of pascal's triangle, we see that

$$\binom{1}{0} = 1,\quad \binom{2}{0} = 1,\quad \binom{3}{0} = 1, \ldots\ldots\ldots\ldots, \quad \binom{n}{0} = 1$$

$$\binom{1}{1} = 1,\quad \binom{2}{2} = 1,\quad \binom{3}{3} = 1, \ldots\ldots\ldots\ldots, \quad \binom{n}{n} = 1$$

We get taking n=5

$$\binom{5}{0} = 1,\quad \binom{5}{1} = 5,\quad \binom{5}{2} = \frac{5\times 4}{1\times 2} = 10$$

$$\binom{5}{3} = \frac{5\times 4\times 3}{1\times 2\times 3} = 10,\quad \binom{5}{4} = \frac{5\times 4\times 3\times 2}{1\times 2\times 3\times 4} = 5$$

and $$\binom{5}{5} = \frac{5\times 4\times 3\times 2\times 1}{1\times 2\times 3\times 4\times 5} = 1$$

Therefore in the case of $\binom{5}{3}$

$$\binom{5}{3} = \frac{5\times(5-1)\times(5-2)}{1\times 2\times 3}$$

and $$\binom{6}{4} = \frac{6\times(6-1)\times(6-2)\times(6-3)}{1\times 2\times 3\times 4}$$

Generaly, we can write-

$$\binom{n}{0} = 1,\quad \binom{n}{n} = 1$$

$$\binom{n}{r} = \frac{n(n-1)(n-2).....(n-r+1)}{1\times 2\times 3\times 4\ldots\ldots\ldots\ldots\times r}$$

Using appropriate sign

$$(1+y)^4 = \binom{4}{0}y^0 + \binom{4}{1}y^1 + \binom{4}{2}y^2 + \binom{4}{3}y^3 + \binom{4}{4}y^4$$

$$= 1 + 4y + 6y^2 + 4y^3 + y^4$$

$$(1+y)^5 = \binom{5}{0}y^0 + \binom{5}{1}y^1 + \binom{5}{2}y^2 + \binom{5}{3}y^3 + \binom{5}{4}y^4 + \binom{5}{5}y^5$$

$$= 1 + 5y + 10y^2 + 10y^3 + 5y^4 + y^5$$

And the expansion of $(1+y)^n$

$$(1+y)^n = \binom{n}{0}y^0 + \binom{n}{1}y^1 + \binom{n}{2}y^2 + \binom{n}{3}y^3 + ... + \binom{n}{n}y^n$$

$$= 1 \cdot y^0 + ny^1 + \frac{n(n-1)}{1 \cdot 2}y^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}y^3 + ... + 1 \cdot y^n$$

i'e the expansion of $(1+ \ y)^n$

$$\therefore (1+y)^n = 1 + ny + \frac{n(n-1)}{1 \cdot 2}y^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}y^3 + ... + y^n.$$

Example 1. Expand $(1+3x)^5$

Solution : By the help of pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$\therefore \ (1+3x)^5 = 1 + 5 \cdot 3x + 10 \cdot (3x)^2 + 10(3x)^3 + 5(3x)^4 + 1(3x)^5$$

$$= 1 + 15x + 90x^2 + 270x^3 + 405x^4 + 243x^5$$

By the help of binomial theorem -

$$(1+3x)^5 = \binom{5}{0}(3x)^o + \binom{5}{1}3x + \binom{5}{2}(3x)^2 + \binom{5}{3}(3x)^3 + \binom{5}{4}(3x)^4 + \binom{5}{5}(3x)^5$$

or, $(1+3x)^5 = 1 + \frac{5}{1}(3x) + \frac{5 \cdot 4}{1 \cdot 2} \cdot (3x)^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}(3x)^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}(3x)^4 + 1 \cdot (3x)^5$

$$= 1 + 15x + 90x^2 + 270x^3 + 405x^4 + 243x^5.$$

Example 1. Expand $(1-3x)^5$

Solution : By the help of pascal's triangle

$$(1-3x)^5 = 1 + 5(-3x) + 10(-3x)^2 + (10(-3x)^3 + 5(-3x)^4 + 1(-3x)^5$$

$$= 1 - 15x + 90x^2 - 270x^3 + 405x^4 - 243x^5.$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

By the help of binomial theorem -

$$(1-3x)^5 = \binom{5}{0}(-3x)^0 + \binom{5}{1}(-3x)^1 + \binom{5}{2}(-3x)^2 + \binom{5}{3}(-3x)^3 + \binom{5}{4}(-3x)^4 + \binom{5}{5}(-3x)^5$$

$$= 1 \cdot 1 + \frac{5}{1} \cdot (-3x) + \frac{5 \cdot 4}{1 \cdot 2}(-3x)^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}(-3x)^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}(-3x)^4 + 1 \cdot (3x)^5$$

$$=1-15x+90x^2-270x^3+405x^4-243x^5.$$

Remark : From the expansion of $(1+3x)^5$ and $(1-3x)^5$, we see that both the expansion are same. Only changing the sign of coefficients we get one from anthor, i'e +, -, +, ……………..

Activity : Expand
$(1+2x^2)^7$ and $(1-2x^2)^7$

Example 3. Expand up to fifth term of $\left(1+\frac{2}{x}\right)^8$

Solution : Do yourself by pascal's triangle
By binomial expansion

$\left(1+\frac{2}{x}\right)^8 = \binom{8}{0}\left(\frac{2}{x}\right)^0 + \binom{8}{1}\left(\frac{2}{x}\right)^1 + \binom{8}{2}\left(\frac{2}{x}\right)^2 + \binom{8}{3}\left(\frac{2}{x}\right)^3 + \binom{8}{3}\binom{2}{4}^4$ .... [up to 5th term]

$$=1\cdot 1+\frac{8}{1}\cdot\frac{2}{x}+\frac{8\cdot 7}{1\cdot 2}\cdot\frac{4}{x^2}+\frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}\cdot\frac{8}{x^3}+\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{16}{x^4}+..................$$

$$=1+\frac{16}{x}+\frac{112}{x^2}+\frac{448}{x^3}+\frac{1120}{x^4}+................$$

$\therefore \left(1+\frac{2}{x}\right)^8 = 1+\frac{16}{x}+\frac{112}{x^2}+\frac{248}{x^3}+\frac{1120}{x^4}+................$ [up to 5th term]

Example 4. Find the coefficients of x^3 and x^6 of $\left(1-\frac{x^2}{4}\right)^8$.

Solution : By binomial expansion

$$\left(1-\frac{x^2}{4}\right)^8 = \binom{8}{0}\left(-\frac{x^2}{4}\right)^0 + \binom{8}{1}\left(-\frac{x^2}{4}\right)^1 + \binom{8}{2}\left(-\frac{x^2}{4}\right)^2 + \binom{8}{3}\left(-\frac{x^2}{4}\right)^3 + \binom{8}{4}\left(-\frac{x^2}{4}\right)^4 + ...$$

$$=1\cdot 1+\frac{8}{1}\cdot\left(-\frac{x^2}{4}\right)+\frac{8\cdot 7}{1\cdot 2}\cdot\frac{x^4}{16}+\frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}\cdot\left(-\frac{x^6}{64}\right)+\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\left(\frac{x^8}{256}\right)+..............$$

$$=1-2x^2+\frac{7}{4}x^4-\frac{7}{8}x^6+...........$$

$\left(1-\frac{x^2}{4}\right)^8$ Here we see that the term containg x^3 is absent. So coefficient of x^3 is 0 and coefficient x^6 is $-\frac{7}{8}$.

$\therefore$ coefficient of x^3 is 0 and coefficient of x^6 is $-\frac{7}{8}$.

Activity : Verify example 4 by the help of pascal's triangle.

Example 5. If we get $1+bx^2$ expanding $(1-x)(1+ax)^6$ upto x^2, find the value of a and b.

Solution : $(1-x)(1+ax)^6$

$$(1-x)\left[\binom{6}{0}\cdot(ax)^0+\binom{6}{1}(ax)^1+\binom{6}{2}(ax)^2+............\right]$$

$$=(1-x)\left[1+\frac{6}{1}\cdot ax+\frac{6\cdot 5}{1\cdot 2}a^2x^2+...........................\right]$$

$$=(1-x)(1+6ax+15a^2x^2+...............)$$

$$=(1+6ax+15a^2x^2+.........)+\ (-x-6ax^2-15a^2x^3-............)$$

$$=1+(6a-1)x+15a^2x^2-6ax^2-15a^2x^3+..................$$

$$=1+(6a-1)x+(15a^2-6a)x^2-15a^2x^3+..................$$

By the question,

$$1+(6a-1)x+(15a^2-6a)x^2-15a^2x^3+..................=1+bx^2$$

$$=1+bx^2$$

Equating the coefficients of x and x^2 from both sides, we get

$$6a-1=0, 15a^2-6a=b$$

or $a=\frac{1}{6}$, and $b=15\cdot\frac{1}{36}-6\cdot\frac{1}{6}=\frac{5}{12}-=-\frac{7}{12}$

$\therefore\ a=\frac{1}{6},\ b=\frac{-7}{12}$

Ans : $a=\frac{1}{6},\ b=\frac{-7}{12}$

Example 6. Find the coefficient of x^7 in the expansion of $(1-x)^8(1+x)^7$.

Solution :

$$(1-x)^8(1+x)^7=(1-x)(1-x)^7(1+x)^7=(1-x)(1-x^2)^7$$

$$=(1-x)\left[\binom{7}{0}(-x^2)^0+\binom{7}{1}(-x^2)^1+\binom{7}{2}(-x^2)^2+\binom{7}{3}(-x^2)^3+\binom{7}{4}(-x^2)^4+.......\right]$$

$$\therefore\ (1-x)^8(1+x)^7=(1-x)[1-7x^2+21x^4-35x^8+35x^6-...........]$$

$$=(1-7x^2+21x^4-35x^6+35x^8-35x^4+............)+(-x+7x^3-21x^5+35x^7-35x^9+..........)$$

$$\therefore\ (1-x)^8(1+x)^7=1-x-7x^2+7x^3+21x^4-21x^5-35x^6+35x^8-...........]$$

$$=(1-x)^8(1+x)^7=1-x-7x^2+7x^3+21x^4-21x^5-35x^6+35x^7+35x^8............$$

$\therefore$ coefficient of x^7 is 35 in the expansin of $(1-x)^8(1+x)^7$

$\therefore$ coefficient of x^7 is 35

Example 7. Expanding $(2-x)\left(1+\frac{1}{2}x\right)^8$ upto x^3 in asending power of *x*. find the value of $1\cdot9\times(1\cdot05)^8$ by using the result.

Solution : Using binomial expansion, we get

$$(2-x)\left(1+\frac{1}{2}x\right)^8=(2-x)\left[\binom{8}{0}\left(\frac{x}{2}\right)^0+\binom{8}{1}\left(\frac{x}{2}\right)^1+\binom{8}{2}\left(\frac{x}{2}\right)^2+\binom{8}{3}\left(\frac{x}{2}\right)^3+\binom{8}{4}\left(\frac{x}{2}\right)^4+.........\right]$$

or $$(2-x)\left(1+\frac{1}{2}x\right)^8=(2-x)\left[1\cdot1+\frac{8}{1}\cdot\frac{x}{2}+\frac{8\cdot7}{1\cdot2}\cdot\frac{x^2}{4}+\frac{8\cdot7\cdot6}{1\cdot2\cdot3}\frac{x^3}{8}+................\right]$$

$$=(2-x)(1+4x+7x^2+7x^3+.............)$$

$$=(2+8x+14x^2+14x^3+.............)$$

$$+(-x-4x^2-7x^3-7x^4-............)$$

$$=2+7x+10x^2+7x^3+.................$$

$$\therefore(2-x)\left(1+\frac{1}{2}x\right)^8=2+7x+10x^2+7x^3+.................$$

$\therefore$ the required expansion $(2-x)\left(1+\frac{1}{2}x\right)^8=2+7x+10x^2+7x^3+.................$

Put $x=0.1$ in the expansion,

$$(2-\cdot1)\times\left(1+\frac{\cdot1}{2}\right)^8=2+7\times(\cdot1)+10(\cdot1)^2+7(\cdot1)^3+...............$$

Or, $1\cdot9\times(1\cdot05)^8=2+\cdot7+10\times(\cdot01)+7(\cdot001)+...............$

Or, $1\cdot9\times(1\cdot05)^8=2+\cdot7+\cdot1+\cdot007+...............$

$=2\cdot807$ (upto three decimal places)

Ans : $1\cdot9\times(1\cdot05)^8=2\cdot807$

Activity : Verify the expansion by pascal's triangle.

Excercise 10·1

1. Find the expansion of $(1+y)^5$ by the help of pascal's triangle or binomial theorem. With the help of the above expansion find $(i)\,(1-y)^5$ and $(ii)\,(1+2x)^5$.
2. According to the asending power of x, expand the following upto first four terms of
   (a) $(1+4x)^6$, $(b)\,(1-3x)^7$.

3. Expand $(1+x^2)^8$ upto first four terms. Find the value of $(1\cdot 01)^8$ by using the result.
4. According to the asending power of x, expand the following upto first three terms of $(a)\,(1-2x)^5$, $(b)\,(1+3x)^9$

   And then expnd, $(c)\,(1-2x)^5(1+3x)^9$ upto x^2.
5. Find the following expansion upto first four terms [using pascal's triangle or binomial theorem]

   $(a)\ (1-2x^2)^7$ $\qquad (b)\ \left(1+\frac{2}{x}\right)^4$ $\qquad (c)\ \left(1-\frac{1}{2x}\right)^7$
6. Expand $(a)\,(1-x)^6$ and $(b)\,(1+2x)^6$ upto x^3 and then expand $(c)\,(1+x-2x^3)^6$ upto x^3 .
7. As x is sufficiently small and neglecting x^3 and higher powers of x, prove that. $(1+x)^5(1-4x)^4 = 1-11x+26x^2$.

10·2 : Binomial expansion of $(x+y)^n$

We discussed the expansion of $(1+y)^n$, now we discuss the general form of binomial expansion where n is positive integer. Generally the expansion of $(x+y)^n$ is known as bionomial theorem.

$$(1+y)^n = 1+\binom{n}{1}y+\binom{n}{2}y^2+\binom{n}{3}y^3+\binom{n}{r}y^r+\ldots\ldots\ldots.\binom{n}{n}y^n$$

Now, $(x+y)^n = \left[x\left(1+\frac{y}{x}\right)\right]^n = x^n\left(1+\frac{y}{x}\right)^n$

$$\therefore\ (x+y)^n = x^n\left[1+\binom{n}{1}\left(\frac{y}{x}\right)+\binom{n}{2}\left(\frac{y}{x}\right)^2+\binom{n}{3}\left(\frac{y}{x}\right)^3+\ldots\ldots\ldots.\binom{n}{n}\left(\frac{y}{x}\right)^n\right]$$

$$\therefore\ (x+y)^n = x^n\left[1+\binom{n}{1}\left(\frac{y}{x}\right)+\binom{n}{2}\frac{y^2}{x^2}+\binom{n}{3}\frac{y^3}{x^3}+\ldots\ldots\ldots.\frac{y^n}{x^n}\right]\ \left[\because\binom{n}{n}=1\right]$$

$$= x^n+\binom{n}{1}\left(x^n\cdot\frac{y}{x}\right)+\binom{n}{2}\left(x^n\cdot\frac{y^2}{x^2}\right)+\binom{n}{3}\left(x^n\cdot\frac{y^3}{x^3}\right)+\ldots\ldots\ldots+.x^n\cdot\frac{y^n}{x^n}$$

$$\therefore (x+y)^n = x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\binom{n}{3}x^{n-3}y^3+\ldots\ldots\ldots.+y^n$$

It is the general form of binomial theorem. Observe it is similar to $(1+y)^n$. Here power of x is added n to 0. We also observe that addition of power of x and y in every term is equal to the power of binomial expansion. Powers of 'x' from starting term to the last is decreasing from n to 0 and conversely the powers of y is increasing from 0 to n.

Example 8. Expand $(x+y)^5$ and from that expansion find $(3+2x)^5$.

Solution : $(x+y)^5 = x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + y^5$

$$= x^5 + 5x^4y + \frac{5\cdot 4}{1\cdot 2}x^3y^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}x^2y^3 + \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}xy^4 + y^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

$\therefore$ The required expasion: $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$

Now put $x = 3$ and $y = 2x$

$$(2+2x)^5 = 3^5 + 5\cdot 3^4\cdot 2x + 10\cdot 3^3(2x)^2 + 10\cdot 3^2(2x)^3 + 5\cdot 3(2x)^4 + (2x)^5.$$

$$= 243 + 810x + 1080x^2 + 720x^3 + 240x^4 + 32x^5$$

There fore, $(3+2x)^5 = 243 + 810x + 1080x^2 + 720x^3 + 240x^4 + 32x^5$

Example 9. Expand $\left(x + \frac{1}{x^2}\right)^6$ upto fourth term in decending power of x and identify the term which is free from ***x*.**

Solution : We get by binomial theorem,

$$\left(x + \frac{1}{x^2}\right)^6 = (x)^6 + \binom{6}{1}x^5\left(\frac{1}{x^2}\right) + \binom{6}{2}x^4\left(\frac{1}{x^2}\right)^2 + \binom{6}{3}x^3\left(\frac{1}{x^2}\right)^3 + \ldots\ldots\ldots\ldots\ldots$$

$$= x^6 + 6x^3 + \frac{6\cdot 5}{1\cdot 2}x^4\cdot\frac{1}{x^4} + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}x^3\,\frac{1}{x^6} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= x^6 + 6x^3 + 15 + 20\frac{1}{x^3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The required expansion $x^6 + 6x^3 + 15 + \frac{20}{x^3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

And term free from x is 15

Example 10. Expand $\left(2 - \frac{x}{2}\right)^7$ upto first four term in asending power of x. Find also $(1\cdot 995)^7$ upto four decimal places.

Solution :

$$\left(2 - \frac{x}{2}\right)^7 = 2^7 + \binom{7}{1}2^6\left(-\frac{x}{2}\right) + \binom{7}{2}2^5\left(-\frac{x}{2}\right)^2 + \binom{7}{3}2^4\left(-\frac{x}{2}\right)^3 + \ldots\ldots\ldots\ldots\ldots$$

$$= 128 + 7\cdot 64\left(-\frac{x}{2}\right) + \frac{7\cdot 6}{1\cdot 2}\cdot 32\cdot\frac{x^2}{4} + \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}\cdot 16\left(-\frac{x}{2}\right)^3 + \ldots\ldots\ldots\ldots$$

$$\therefore \left(2 - \frac{x}{2}\right)^7 = 128 - 224x + 168x^2 - 70x^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\therefore$ The required expasion $\left(2-\frac{x}{2}\right)^7 = 128-224x+168x^2-70x^3+......................$

Now, $2-\frac{x}{2}=1\cdot 995$

Or, $\frac{x}{2}=2\cdot 000-1\cdot 995$

Or, $x=0\cdot 01$ (putting)

$$\left(2-\frac{0\cdot 01}{2}\right)^7 = 128-224\times(\cdot 01)^2+168\times(\cdot 01)^2-70\times(\cdot 01)^2+......................$$

Or, $(1\cdot 995)^7 = 125\cdot 7767$ (upto four decimal places).

10·3. Find the value of $n\,!$ and n_{c_r}

Observe the following exanples

$2=2\cdot 1$
$6=3\cdot 2\cdot 1$
$24=4\cdot 3\cdot 2\cdot 1$
$120=5\cdot 4\cdot 3\cdot 2\cdot 1$
$-----------$
$-----------$

We can express briefly the product of right side by symbol

$2=2\cdot 1=2\,!$
$6=3\cdot 2\cdot 1=3\,!$
$24=4\cdot 3\cdot 2\cdot 1=4\,!$
$120=5\cdot 4\cdot 3\cdot 2\cdot 1=5\,!$
$-------------$
$-------------$

Now we observe

$4\,!=4\cdot 3\cdot 2\cdot 1$
$\quad =4\cdot(4-1)\cdot(4-2)\cdot(4-3)$
$5\,!=5\cdot 4\cdot 3\cdot 2\cdot 1$
$\quad =5\cdot(5-4)\cdot(5-2)(5-3)\cdot(5-4)$
$-------------------$
$-------------------$

$\therefore$ Generally, we can write $n\,!=n(n-1)(n-2)(n-3).........3\cdot 2\cdot 1$

and $n\,!$ is called factorial of n

Similarly, $3\,!$ is called factorial of 3,

$4\,!$ is called factorial of 4

Again we observe :

$$\binom{5}{3}=\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}=\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{(1\cdot 2\cdot 3)\cdot(2\cdot 1)}$$

$$= \frac{5!}{3!\times 2!} = \frac{5!}{3!(5-3)!}$$

$$\binom{7}{4} = \frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4} = \frac{7\cdot 6\cdot 5\cdot 4}{4!}$$

$$= \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4!\times(3\cdot 2\cdot 1)} = \frac{7\ !}{4!\times 3\ !}$$

$$= \frac{7!}{4!(7-4)!}$$

$\therefore$ Generally we can say, $\binom{n}{c} = \frac{n!}{r!(n-r)!}$

Factorial of right side can be expressed by the symbol following.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = n_{c_r}$$

$$\therefore \quad \binom{7}{4} = \frac{7!}{4!(7-4)!} = 7_{c_4}$$

and $$\binom{5}{3} = \frac{5!}{3!(5-3)!} = 5_{c_3}$$

Therefore, $\binom{n}{r} = n_{c_r}$

That is, $\binom{n}{r}$ and n_{c_r} are equal.

$$\therefore \binom{n}{1} = n_{c_1}, \quad \binom{n}{2} = n_{c_2}$$

$$\binom{n}{3} = n_{c_3)} \ .... \ \binom{n}{n} = n_{c_n}$$

We know $\binom{n}{n} = 1 = n_{c_n}$

Now $n_{c_n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!(b)!} = \frac{1}{0!}$

$\therefore\ 1 = \frac{1}{0!}$

That is, $0! = 1$.

remember

$\therefore$

$n! = n(n-1)(n-z)(n-3)$................3.2.1
$\binom{n}{r} = n_{c_r}$, $\quad n_{c_n} = 1$,
$\binom{n}{r} = n_{c_r} = \frac{n!}{r!(n-r)!}$, $\binom{n}{o} = n_{c_0} = 1$
$\binom{n}{n} = n_{c_n} = 1,\ 0! = 1.$

Now we use n_{c_r} instead of $\binom{n}{r}$ in binomial theorem

$$(1+y)^n = 1 + n_{c_1} y + n_{c_2} y^2 + n_{c_3} y^3 + ...+ n_{c_r} y^r + ...n_{c_n} y^n$$

Or,

$$(1+y)^n = 1 + ny + \frac{\eta(n-1)}{2!} y^2 + \frac{n(n-1)(n-2)}{3!} y^3 + \frac{n(n-1)(n-2)(n-3)}{4!} y^4 .... + y^n$$

That is $(1+y)^n = 1 + ny + \frac{\eta(n-1)}{1\cdot 2} y^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} y^3 + ... + y^n$

And similarly,

$$(x+y)^n = x^n + n_{c_1} x^{n-1} y + n_{c_2} x^{n-2} y^2 + n_{c_3} x^{n-3} y^3 + ...+ n_{c_r} x^{n-r} y^r + ...n_{c_n} y^n$$

Or, $(x+y)^n = x^n + nx^{n-1} y + \frac{\eta(n-1)}{2!} x^{n-2} y^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3} y^3 + ... + y^n$

Observe : For positive integer n

1. Here the general term or r-th term of $(1+y)^n$ is r $T_{r+1} = \left(\frac{n}{r}\right) y^r$ or $n_{c_r} y^r$

Now, $\left(\frac{n}{r}\right)$ or n_{c_r} is the coefficient of binomial expansion

$$(x+y)^n = x^n + n_{c_1} x^{n-1} y + n_{c_2} x^{n-2} y^2 + n_{c_3} x^{n-3} y^3 + n_{c_4} x^{n-4} y^4 + ...+ y^n$$

$$= x^n + nx^{n-1} y + \frac{\eta(n-1)}{2!} x^{n-2} y^2 + \frac{n(n-1)(n-2)}{3!} x^{n-3} y^3 + \frac{n(n-1)(n-2)(n-3)}{4!} x^{n-4} y^4 .... + y^n$$

The general term or r-th term

$$T_{r+1} = \left(\frac{n}{r}\right) x^{n-r} y^r \text{ or } n_{c_r} x^{n-r} y^r$$

Where $\left(\frac{n}{r}\right)$ or n_{c_r} is the coefficient of binomial expansion

Example 11. Expand $\left(x-\frac{1}{x^2}\right)^5$

Solution : By binomial theorem

$$\left(x-\frac{1}{x^2}\right)^5 = x^5 + 5_{c_1}x^{5-1}\left(-\frac{1}{x^2}\right)+5_{c_2}x^{5-2}\left(-\frac{1}{x^2}\right)^2+5_{c_3}x^{5-3}\left(-\frac{1}{x^2}\right)^3+5_{c_4}x^{5-4}\left(-\frac{1}{x^2}\right)^4+\left(-\frac{1}{x^2}\right)^5$$

$$= x^5 - 5x^4\cdot\frac{1}{x^2}+\frac{5\cdot4}{1\cdot2}x^3\cdot\frac{1}{x^4}-\frac{5\cdot4\cdot3}{1\cdot2\cdot3}x^2\cdot\left(\frac{1}{x^6}\right)+\frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}x\cdot\left(\frac{1}{x^8}\right)-\frac{1}{x^{10}}$$

$$= x^5 - 5x^3 + \frac{10}{x}-\frac{10}{x^4}+\frac{5}{x^7}-\frac{1}{x^{10}}$$

Example 12. Expand $\left(2x^2-\frac{1}{x^2}\right)^8$ upto four terms

Solution :

$$\left(2x^2-\frac{1}{x^2}\right)^8 = (2x^2)^8 + 8_{c_1}(2x^2)^7\left(-\frac{1}{2x}\right)+8_{c_2}(2x^2)^6\left(-\frac{1}{2x}\right)^2+8_{c_3}(2x^2)^5\left(-\frac{1}{2x}\right)^4+\text{..........}$$

$$= 256x^{16}-512x^{13}+448x^{10}-224x^7+\text{...........}$$

Exercise 10·2

1. i $8_{c_0} = 8_{c_8}$

   ii $\left(\frac{n}{r}\right) = \frac{n(n-1)(n-2)......(n-r+1)}{r}$

   iii In the expansion of $(1+x)^n$, the 2nd term is $= \frac{n(n-1)}{<2}x^2$

   Which is correct of the following ?

   a. i, ii b. ii, iii
   c. i, iii d. i, ii, iii

2. The expansion $(a+x)^n$ has (n+1) terms. Where n is

   a. Non negative number b. Positive number
   c. Negative number d. Fraction

3. The coefficients of expansion $(x+y)^5$ are

   a. 5, 10, 10, 5 b. 1, 5, 10, 10, 5, 1
   c. 10, 5, 5, 10 d. 1, 2, 3, 3, 2, 1

4. The coefficients of x in the expansion $(1-x)\ (1+\frac{x}{2})^8$ are

   a. -1 b. $\frac{1}{2}$ c. 3 d. $-\frac{1}{2}$

5. x free term of the expansion of $(x^2+\frac{1}{x^2})^4$ is.

a. 4 b. 6 c. 8 d. 0

6. If we get $+9x+cx^2$ expanding $(2-x)(1+ax)^5$ upto x^2, find *a* and *c*

a. a = 1, c = 15 b. a = 5, c = 15

c. a = 15 c = 1 d. a = 1, c = o

Answer question no: 7 & 8 in the light of following informations

If $^n c_r = \frac{n!}{r!(n-r)!}$

7. $^n c_0$ = Whit?

a. 0 b. 1 c. n d. Undetermined

8. If n = r = 100, Find the value of $^n c_r$

a. 0 b. 1 c. 100 d. 200

9. We get arranging the coefficients of $(x+y)^4$

a.

```
      4
    1  4  1
  1  5  5  1
 1  6  10  6  1
```

b.

```
      1
    1  2  1
  1  3  3  1
 1  4  6  4  1
```

c.

```
      2
    2  3  2
  1  5  5  2
 2  7  10  7  2
```

d.

```
       6
     6  12  6
   6  18  18  6
 6  24  36  24  6
```

10. Expand the following :

(*a*) $(2+x^2)^5$ (*b*) $\left(2-\frac{1}{2x}\right)^6$

11. Find the first four terms of the following expansion

(*a*) $(2+3x)^6$ (*b*) $\left(4-\frac{1}{2x}\right)^5$

12. If $\left(p-\frac{1}{2}x\right)^6 = r-96x+5x^2+.........$, Find the value of, *p* and *r*

13. Find the coefficient of x^3 in the expansions of

14. Expand $\left(2+\frac{x}{4}\right)^8$ upto x^3 in asending power of *x*. Find the value of $(1\cdot9975)^6$ upto four decimal places

15. Using binomial theorem, find the value of $(1\cdot99)^5$ upto four decimal places

16. In the expansion of $\left(1+\frac{1}{4}\right)^n$, coefficients of 3rd term is the double of the coefficient of the 4th term. Find the value of n.

17. (*a*) In the expansion of $\left(k-\frac{x}{3}\right)^7$ coefficients of k^3 is 560, find the value of x.

(*b*) In the expansion of $\left(k^2+\frac{k}{x}\right)^6$ coefficients of x^3 is 160, find the value of k.

18. Given,

$P = (a+bx)^6$ ..... (i)
$Q = (b+ax)^5$ ..... (ii)
$R = (a+x)^n$ ..... (iii)

a. Write down the expansion of (iii) and applying the formula, find the expansion of (i)
b. If the ratio of the 2nd and 3rd term of (i) and ratio of the 2nd and 3rd term of (ii) are equal, so that a:b= $\sqrt{5}$: 2, give an example in the light of the above statement.
c. Show that the summation of the absolute constant of the even term is equal to the summation of the absolute constant of the odd term. Mention a binomial expression for which the above statement is true.

Chapter Eleven
Coordinate Geometry

The portion of geometry where the algebraic expressions of points, straight lines and curved lines are studied is familiar as the Coordinate Geometry. This portion of geometry is also known as the Analytic Geometry. With the plotting of points on the plane, the straight lines or the curved lines or the figures of the geometric regions made by them such as the triangles, the quadrilaterals, the circles etc. are expressed. The system of plotting of points on the plane is initiated by a French mathematician Rene Descates (known as Descartes). The coordinate system of geometry initiated by Descartes is called after his name the Cartesian coordinate system. The coordinate geometry and the analytical geometry are mainly based on Cartesian coordinate system. So, Descates is called the initiator of the analytical geometry.
In the first part of the chapter, through developing the concept of Cartesian coordinate system among the learners, the tricks of determining the distance between the two points will be discussed. In the second part, the method of determining the measurement of any triangle, quadrilateral created by the straight line will be discussed and in the third part, the tricks of determinining the slope of the straight line and determining the simplification of the joining straight line between the two points will be explained. No figure or equation associated with the curved line will be discussed here. In higher classes, it will be discussed elaborately.

After completing this chapter, the students will be able to –
- Explain the rectangular Cartesian coordinate system
- Find the distance between two points
- Explain the concept of slope (gradient) of a straight line
- Find the equation of a straight line
- Find areas of triangular and quadrangular regions by measuring the lengths of the sides
- Construct the geometric figures of the triangles and the quadrilateral by plotting of points
- Present the equation by plotting of points.

11·1 Rectangular Cartesian Coordinates
We have been acquainted with the concept of plane in previous class. The surface of a table, the floor of a room, the surface of a book, even the paper on which we write down, each of them is a plane. The surface of a football or the surface of a bottle is a curved plane. In this part, the tricks of determining
the proper position of any point lying on the plane will be discussed. For determining the proper position of any definite point, it is necessary to know the distance of the

definite point from the straight line bisectors constructed on the plane. It is said as a reason that only a point can lie at any definite distance from the two straight line bisectors.

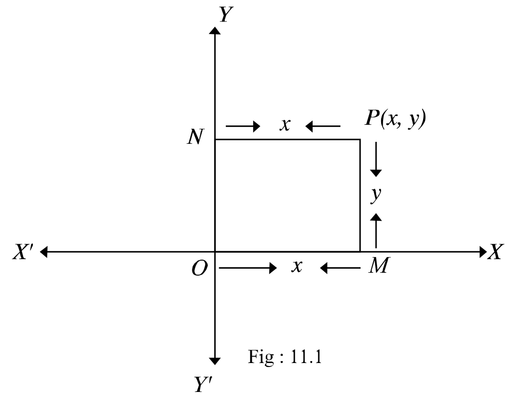


Fig : 11.1

If two such straight lines XOX' and YOY' are drawn that intersect each other at the right angle on any plane, XOX' is called the *x-axis*, YOY' is called the *y-axis* and the intersecting point *'O'* is called the *Origin*

Now, let P be any point on the plane of the two axes. From that P, on XOX' i.e., the *x-axis* and on YOY' i.e., the *y-axis*, the perpendiculars respectively PM and PN. Then the distance of P from the *y-axis* $= NP = OM = x$ is called the *abscissa* of P or *x-coordinate*. Again the distance of P from the *x-axis* $= MP = ON = y$ is called the *Ordinate* of P or *y-coordinate*. The *abscissa* and the *Ordinate* are jointly called the *Coordinate*. So, in the figure, the coordinate of P means the perpendicular distance of P from the *y-axis* and the *x-axis* and by donoting them as x and y, the coordinate of P is expressed by the symbol $P(x,y)$.

The coordinate index (x,y) means an ordered pair whose first one indicates the *abscissa* and the second one indicates the *Ordinate*. So, by (x,y) and (y,x), two different points are meant. Therefore, the coordinate of any point depending on two axes intersecting each other at the right angle is called the *Rectangular Cartesian Coordinates*. If the point is placed at the right side of the *y-axis*, the abscissa will be positive and if it is placed at the left side, the abscissa will be negative. Again, if the point is placed above the *x-axis*, the ordinate will be positive and if it is placed below, the ordinate will be negative. On the *x-axis*, the ordinate will be zero and on *y-axis*, the abscissa will be zero.

So, the abscissa and the ordinate of any point will remain through OX and OY respectively or parallel to them. Similarly, the negative abscissa or the ordinate will remain through OX and OY which are parallel to them.

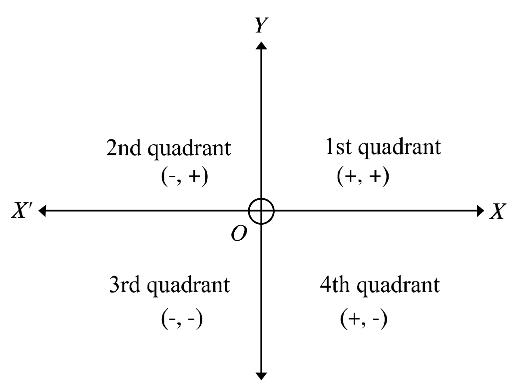


Fig : 11.2

By the two axes of the *Cartesian Coordinates*, the plane is divided into XOY, YOX', $X'OY'$, $Y'OX$ these four parts. Each of them is called the *Quadrant*.

The Quadrant XOY is taken as the first and by turns the second, the third, the fourth quadrant remain in anti-clockwise order. As per the sign of of the point of the coordinate, the point lies on the different quadrant.

11·2 Distance between two Points

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two different points on a plane. From P and Q, draw perpendiculars PM and QN on the x-*axis*. Again, from P, draw perpendicular PR on QN. Now the abscissa of $P = OM = x_1$

And the ordinate of $P = MP = y_1$

The abscissa of $Q = ON = x_2$ and the ordinate $NQ = y_2$

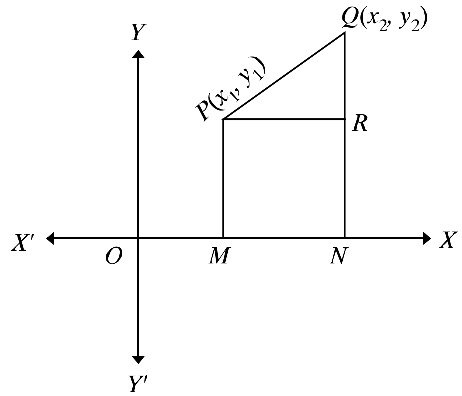


Fig : 11.3

From the figure we get,

$$PR = MN = ON - OM = x_2 - x_1$$

$$QR = NQ - NR = NQ - MP = y_2 - y_1$$

As per the construction, PQR is a right angled triangle and PQ is the hypotenuse of the triangle. So, as per the theorem of Pythagoras,

$$PQ^2 = PR^2 + QR^2$$

$$\text{or, } PQ = \pm\sqrt{PR^2 + QR^2}$$

$$\therefore PQ = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance of P from Q

$$\therefore PQ = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

As the distance is always non-negative, the negative value has been avoided.

Again, in the same rule $QP = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ the distance of P from Q

$\therefore PQ = QP$

The distance of Q from P or P from Q is the same.

Corollary : The distance of any point $P(x, y)$ lying on the plane from the origin (0,0) is

$$PQ = \sqrt{(x - 0)^2 + (y - 0)^2}$$

$$OP = \sqrt{x^2 + y^2}$$

Example 1. Plot the two points (1, 1) and (2, 2). Find the distance between them.

Solution : Let $P(1, 1)$ and $Q(2, 2)$ be the given two points.

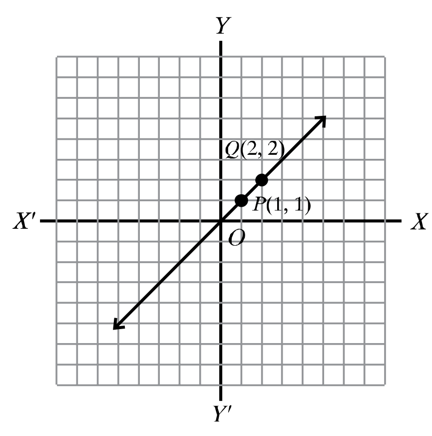


Fig : 11.4

In the figure, the two points have been plotted on the plane xy.

The distance between the two points $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore \quad = \sqrt{(2-1)^2 + (2-1)^2}$ unit.

$= \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$ unit.

Example 2. Plot the origin $O(0,0)$ and the other two points $P(3,0)$ and $Q(0,3)$ on the plane. Find the distance between each of them. What is the name of the geometric figure after joining these points? And why?

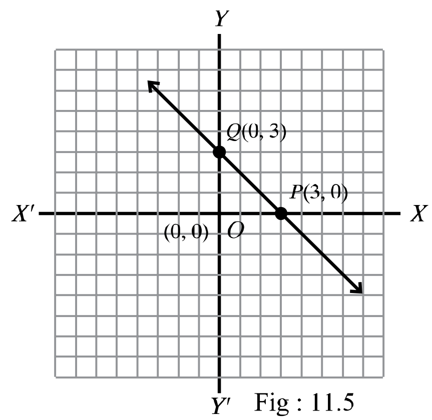


Fig : 11.5

Solution : The position of the three points $O(0,0), P(3,0)$ and $Q(0,3)$ has been shown on the plane xy.

Distance $OP = \sqrt{3^2 + 0^2} = \sqrt{3^2} = 3$ unit.

Distance $OQ = \sqrt{0^2 + 3^2} = \sqrt{3^2} = 3$ unit.

Distance $PQ = \sqrt{(3-O)^2 + (O-3)^2}$ unit.

$= \sqrt{y^2 + y^2} = \sqrt{9+9}$ unit.

$= \sqrt{18} = \sqrt{2 \cdot 9} = 3\sqrt{2}.$

The name of the geometric figure is the isosceles triangle because the distance between the two sides OP and OQ is equal.

Example 3. The three vertices of a triangle are respectively $A(2,0)$, $B(7,0)$ and $C(3,4)$. Plot these points on the plane and construct the triangle. Find the perimeter of the triangle upto five places of decimals.

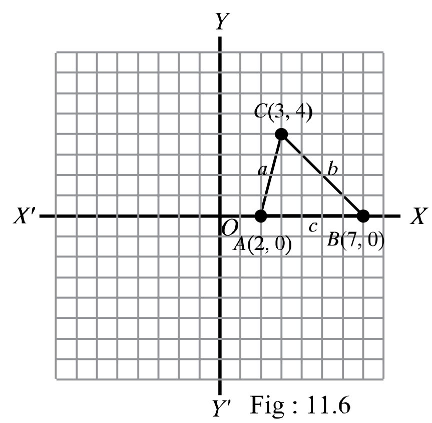


Fig : 11.6

Solution : The position of $A(2,0)$, $B(7,0)$ and $C(3,4)$ on the plane xy has been shown.

Of the triangle

The length of the side $AB(c) = \sqrt{(7-2)^2 + (0-0)^2} = \sqrt{5^2} = 5$ unit

The length of the side $BC\,(a) = \sqrt{(3-7)^2 + (4-0)^2} = \sqrt{(-4)^2 + 4^2} = \sqrt{16+16} = 4\sqrt{2}$ unit

The length of the side $AC\,(b) = \sqrt{(3-2)^2 + (4-0)^2} = \sqrt{1^2 + 4^2} = \sqrt{17}$ unit.

$\therefore$ Perimeter of the triangle $= (AB + BC + AC)$

$= (a + b + c)$

$= (5 + 4\sqrt{2} + \sqrt{17})$ unit $= 14 \cdot 77996$ unit (app.)

Example 4. Show that the points $(0,-1)$, $(-2,3)$, $(6,7)$ and $(8,3)$ are the vertices of a rectangle.

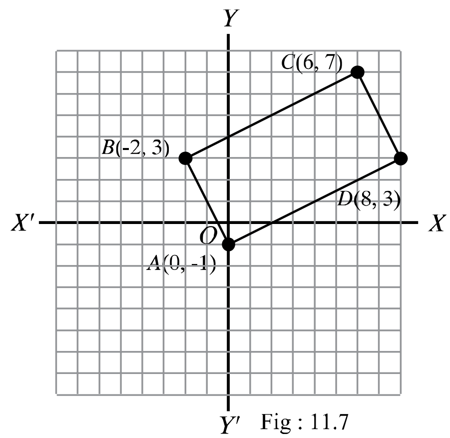


Fig : 11.7

Solution : Let $A(0,-1)$, $B(-2,3)$, $C(6,7)$ and $D(8,3)$ be the given points. Their position on the plane *xy* has been shown.

The length of the side

$AB=\sqrt{(-2-0)^2+\{3-(-1)\}^2}=\sqrt{(-2)^2+(4)^2}=\sqrt{4+16}=\sqrt{20}=2\sqrt{5}$ unit.

The length of the side $CD=\sqrt{(8-6)^2+(3-7)^2}=\sqrt{4+16}=\sqrt{20}=2\sqrt{5}$ unit.

$\therefore$ The length of the side AB = The length of the side CD

Again,

The length of the side $AD=\sqrt{(8-0)^2+(3-(-1))^2}=\sqrt{8^2+4^2}=\sqrt{80}=4\sqrt{5}$ unit.

The length of the side $BC=\sqrt{(6-(-2))^2+(7-3)^2}=\sqrt{8^2+4^2}=\sqrt{80}=4\sqrt{5}$ unit.

$\therefore$ The length of the side AD = The length of the side BC

$\therefore$ The lengths of the opposite sides are equal.

Therefore, we can say *ABCD* is a parallelogram or a rectangle.

The length of the diagonal *BD*

$=\sqrt{(8-(-2))^2+(3-3)^2}=\sqrt{(10)^2+(0)^2}=\sqrt{100}=10=100=10$ unit

Now, $BD=100$, $AB=(2\sqrt{5})^2=20$, $AD=(4\sqrt{5})^2=80$

$\therefore BD^2=AB^2+AD^2=20+80=100$

$\therefore BD^2=AB^2+AD^2$

$\therefore$ As per the theorem of Pythagoras, *ABC* is a right angled triangle and $\angle BAD$ is a right angle.

So, it is proved that *ABCD* is a rectangle.

Example 5. Show that the three points $(-3,-3)$, $(0,0)$ and $(3,3)$ do not form a triangle.

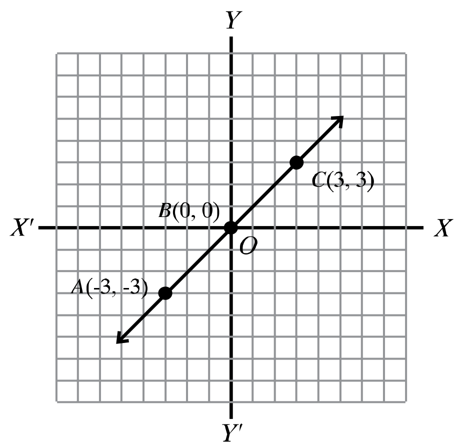


Fig : 11.8

Solution : Let $A(-3,-3)$, $B(0,0)$ and $C(3,3)$ be the given three points. Their position on the plane *xy* has been shown.

We know, the sum of the two sides of any triangle is greater than its third side. Let *ABC* be a triangle and *AB, BC* and *AC* be its there sides.

Now, the length of the side $AB = \sqrt{\{0-(-3)\}^2+\{0-(-3)\}^2} = \sqrt{3^2+3^2} = \sqrt{18} = 3\sqrt{2}$ unit

The length of the side $BC = \sqrt{(3-0)^2+(3-0)^2} = \sqrt{18} = 3\sqrt{2}$ unit

The length of the side $AC = \sqrt{(3+3)^2+(3+3)^2} = \sqrt{72} = 6\sqrt{2}$ unit

It is seen, $AB+BC = 3\sqrt{2}+3\sqrt{2} = 6\sqrt{2} = AC$ figure

i.e., the sum of the two sides is equal to the third side.

$\therefore$ The three points lie on a straight line and it is not possible to form a triangle with them.

Exercise 11·1

1. Find the distance between the given points in every case :
   (i) $(2,3)$ and $(4,6)$ (ii) $(-3,7)$ and $(-7,3)$
   (iii) (a,b) and (b,a)
   (iv) $(0,0)$ and $(\sin\theta,\cos\theta)$ (v) $\left(-\frac{3}{2},-1\right)$ and $\left(\frac{1}{2},2\right)$
2. The three vertices of a triangle are $A(2,-4)$, $B(-4,4)$ and $C(3,3)$. Draw the triangle and show that it is an isosceles triangle.
3. $A(2,5)$, $B(-1,1)$ and $C(2,1)$ are the three vertices of a triangle. Draw the triangle and show that it is a right angled triangle.

4. Ascertain whether the points $A(1, 2)$, $B(-3, 5)$ and $C(5, -1)$ form a triangle.
5. If the two points $(-5, 5)$ and $(5, k)$ are equidistant from the origin; find the value of k.
6. Show that $A(2, 2)$, $B(-2, -2)$ and $C(-2\sqrt{3}, 2\sqrt{3})$ are the vertices of an equilateral triangle. Find its perimeter upto three places of decimals.
7. Show that the points $A(-5, 0)$, $B(5, 0)$, $C(5, 5)$ and $D(-5, 5)$ are the four vertices of a square.
8. Ascertain whether the quadrilateral formed with the points $A(-2, -1)$, $B(5, 4)$, $C(6, 7)$ and $D(-1, 2)$ is a parallelogram or a rectangle.
9. Which of the points $A(10, 5)$, $B(7, 6)$, $C(-3, 5)$ is the nearest to the point $P(3, -2)$ and which is the farthest?
10. From the point $P(x, y)$, the distance of the y-axis is equal to the distance of P from $Q(3, 2)$. Prove that $y^2 - 4y - 6x + 13 = 0$.

11·3 Area of Triangles

We know, if we connect straight lines with three different points that are not lying on the same straight line, we get the region of a triangle. That region of the triangle may be different respective of the side and the angle. In this part we will be able to determine the area of the triangle by finding the sides of any triangle with the help of only one formula. With the help of the same formula by dividing any quadrilateral into the two triangles, the determination of the area of the region of the quadrilateral will be possible. In this case we will determine the area by the perimeter of the triangle (the sum of the lengths of the sides) and and the length of the side. To determine the area of triangle shaped or angle shaped land, the method i.e. by the length of side, is very important. That is, it is very useful in determining the area of the land. It is said as reason, if the coordinates of the vertices of the triangular or quadrangular land are not known or are not possible to know but if the coordinates are known, we will be able to determine the area more easily.In this part we shall determine the area of triangle or polygon by these two methods.

Method 1:

The determination of the Area: A triangle *ABC* has been shown in the adjacent figure. $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the three different points and *AB, BC* and *CA* are the three sides of the triangle. With the help of the formula of determining distance, it is possible to determine easily the lengths of the sides *AB, BC* and *CA.* For example,

Taking *'c'* the length of *AB*, $c = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ unit

Taking *'a'* the length of *BC*, $a = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$ unit

Taking *'b'* the length of *AC*, $c = \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}$ unit

Now, taking *'2s'* as the perimeter of the triangle, $2s = a + b + c$ [the perimeter = the sum of the lengths of the three sides.]

i.e., $s = \frac{1}{2}(a+b+c)$ unit.

Here *s* is the half of the perimeter of the triangle.

We can determine easily the area of any triangle with the help of *s* and *a,b,c*.

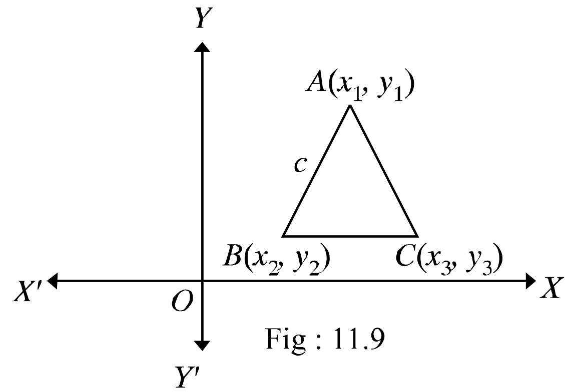


Fig : 11.9

The Formula of Determining the Area of Triangular Region

The length of the side *AB* is '*c'*, the length of the side *BC* is '*a'* and the length of the side *CA* is '*b'* and the perimeter *2s* of the triangle *ABC*, the area of Δ*ABC* is $\sqrt{s(s-a)(s-b)(s-c)}$ sq. unit. [The proof has been given in the part of mensuration of the geometry of Mathematics in Secondary level. The learners will see the proof.]

By the following examples, the use of the formula will be easily understood.

Observed: There are different formulae for determining the area of different triangles but here we shall able to determine the area of any triangle with the help of only one theory.]

Example 1 . The points *A* (2, 5), *B*(-1, 1) and *C*(2, 1) are the three vertices of a triangle. Draw a rough figure of the triangle and find its area by the perimeter and the length of the side. What type of a triangle does it appear to be ? Justify your contention.

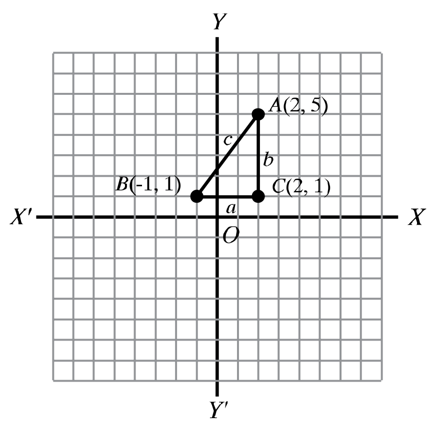


Fig : 11.10

Solution : The triangle has been shown in the adjacent figure.

The length of the side *AB* ,

$c = \sqrt{(-1+2)^2 + (1-5)^2}^{\,2} = \sqrt{9+16} = 5$ unit

The length of the side *BC* ,

$a = \sqrt{(2+1)^2 + (1-1)}^{\,2} = \sqrt{9+0} = 3$ unit

The length of the side AC , $b = \sqrt{(2-2)^2 + (1-5)}^{\,2} = \sqrt{0+16} = 4$ unit

$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(5+3+4) = 6$ unit.

$\therefore$ Area $= \sqrt{s(s-a)(s-b)(s-c)}$ Sq. unit

$= \sqrt{6(6-5)(6-3)(6-4)}$

$= \sqrt{6 \cdot 1 \cdot 3 \cdot 2}$ Sq. unit

$= 6$ Sq. unit

From the figure wc can understand that it is a right angled triangle.
It is proved easily by the theroem of Pytagoras.

$AB^2 = c^2 = 5^2 = 25$ sq. unit
$BC^2 = a^2 = 3^2 = 9$

$CA^2 = b^2 = 4^2 = 16$ sq. unit

$\therefore \; AB^2 = 25 = BC^2 + CA^2 = 9 + 16 = 25$ sq. unit

$\therefore \; ABC$ is a right angle triangle. AB is a hypotenuse and ACB is a right angle.

Example 2. $A(2,-4)$, $B(-4,4)$ and $C(3,3)$ are the three vertices of a triangle. Draw the triangle and find the area by determining the length of the side. What type of a triangle does it appear to be? Justify your contention.

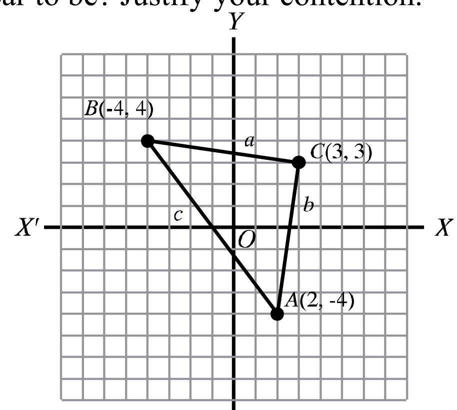


Fig : 11.11

Solution : The triangle ABC is drawn in the figure.
We have,

$AB = c = \sqrt{(-4-2)^2 + \{4-(-4)\}^2} = \sqrt{36+64} = \sqrt{100} = 10$ unit

$BC = a = \sqrt{(3-(-4))^2 + (3-4)^2} = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$ unit

$CA = b = \sqrt{(2-3)^2 + (-4-3)^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$ unit

Now,

$\therefore \; s = \frac{1}{2}(a+b+c) = \frac{1}{2}(10 + 5\sqrt{2} + 5\sqrt{2}) = \frac{1}{2}(10 + 10\sqrt{2} = 5 + 5\sqrt{2}$ unit

$\therefore$ Area, $= \sqrt{s(s-a)(s-b)(s-c)}$ Sq. unit

$$= \sqrt{(5+5\sqrt{2})(5+5\sqrt{2}-10)(5+5\sqrt{2})-5\sqrt{2})(5+5\sqrt{2}-5\sqrt{2})} \text{ Sq. unit}$$

$$= \sqrt{(5+5\sqrt{2})(5\sqrt{2})-5)\cdot 5\cdot 5} \text{ Sq. unit}$$

$$= 5\sqrt{(5+\sqrt{2}+5)(5\sqrt{2}-5)} \text{ Sq. unit}$$

$$= 5\sqrt{(5\sqrt{2})^2-5^2} = 5\sqrt{50-25} = 5\sqrt{25} \text{ Sq. unit}$$

$$= 5\cdot 5 = 25 \text{ Sq. unit}$$

The given triangle is an isosceles triangle. Because, $BC = CA = 5\ \sqrt{2}$ unit. i.e., the two sides of the triangle is equal.

Again, $AB^2 = 10^2 = 100$

$$BC^2 + CA^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 = 50 + 50 = 100$$

$\therefore \quad AB^2 = BC^2 + CA^2$

$\therefore$ This is a right angled triangle.

i.e., ABC is a right angled and isosceles triangle.

Example 3. The three vertices of a triangle are respectively $A(-2,0)$, $B(5,0)$ and $C(1,4)$. Find the length of each side and the area of the triangle. What type of a triangle is it? Justify your contention.

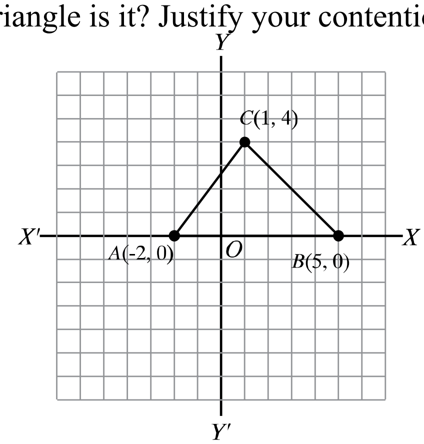


Fig : 11.12

Solution : The triangle is shown in the figure (figure 11.12).

We have,

$$AB = c = \sqrt{(5-(-2))^2 + (0-0))^2} = \sqrt{49} = 7 \text{ unit}$$

$$BC = a = \sqrt{(1-5)^2 + (4-0)^2} = \sqrt{16+16} = 4\sqrt{2} \text{ unit}$$

$$CA = b = \sqrt{(-2-1)^2 + (0-4)^2} = \sqrt{9+16} = 5 \text{ unit}$$

$$\therefore\ S = \frac{1}{2}(a+b+c) = \frac{1}{2}(7+4\sqrt{2}+5) = \frac{1}{2}(12+4\sqrt{2}) = 6+2\sqrt{2} \text{ unit}$$

$\therefore$ Area = $\sqrt{s(s-a)(s-b)(s-c)}$ Sq. unit

$= \sqrt{(6+2\sqrt{2})(6+2\sqrt{2}-7)(6+2\sqrt{2})-4\sqrt{2}))(6+2\sqrt{2}-5)}$ Sq. unit

$= \sqrt{(6+2\sqrt{2})(2\sqrt{2}-1)(6-2\sqrt{2})(2\sqrt{2}+1)}$ Sq. unit

$= \sqrt{(6+2\sqrt{2})(6-2\sqrt{2}+)(2\sqrt{2}+1)(2\sqrt{2}-1)}$ Sq. unit

$= \sqrt{(6)-(2\sqrt{2})^2)((2\sqrt{2})^2-1^2)} = \sqrt{28\cdot 7} = 14$ Sq. unit

The given triangle is an obtuse angled triangle. Because, it has no side equal to its any other side.

Observed: The three triangular regions whose area we have determined are its first one a right angled triangle, the second one an isosceles triangle and the third one an obtuse angled triangle. The area of each triangle has been determined with the help of only one formula. In the same way we can determine the area of any other triangle. In the exercise, the problems related to the area of more such type of triangle will be placed.

In this stage we shall discuss the tricks of determining the area of the quadrilateral by using the same formula.

Example 1. The points $A(1,0)$, $B(0,1)$, $C(-1,0)$ and $D(0,-1)$ are respectively the four vertices of a quadrilateral. Draw the quadrilateral and find the area by determining its lenght of any two sides and the diagonals.

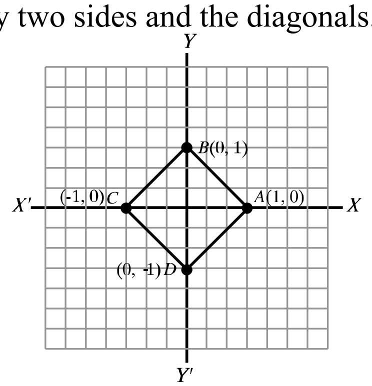


Fig : 11.13

Solution: In the adjacent figure by plotting the points, the quadrilateral $ABCD$ has been shown. *AB, BC, CD* and *DA* are the four sides of the quadrilateral and *AC* and *BD* are the two diagonals of the quadrilateral.

Side $AB = c = \sqrt{(1-0)^2+(0-1)}^2 = \sqrt{13+17} = \sqrt{2}$ unit

Side $BC = a = \sqrt{(0+1)^2+(1-0))}^2 = \sqrt{1^2+1^2} = \sqrt{2}$ unit

Diagonal $AC = b = \sqrt{(1+1)^2+(0-0)}^2 = \sqrt{2^2} = 2$ unit

$\therefore \quad AC^2 = 4$

Side $CD = \sqrt{(-1-0)^2 + (0-1)^2} = \sqrt{2}$ unit

side $DA = \sqrt{(0-1)^2 + (-1-0)^2} = \sqrt{2}$ unit

It is seen, $AB = BC = CD = DA = \sqrt{2}$ unit

$\therefore$ The quadrilateral is a squae or a rhombus.

Now $AC^2 = AB^2 + BC^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 2 + 2 = 4$

$\therefore$ The quadrilateral is a square.

The area of the quadrilateral *ABCD* = 2 × the area of the triangle *ABC*

Now the perimeter of the triangle *ABC,*

$2s = AB + AB + BC = 2 + \sqrt{2} + \sqrt{2} = 2 + 2\sqrt{2}$ unit

$s = \frac{1}{2}(2 + 2\sqrt{2}\,) = 1 + \sqrt{2}$ unit

$\therefore$ The area of the triangle *ABC*

$= \sqrt{s(s-a)(s-b)(s-c)}$ Sq. unit

$= \sqrt{(1+\sqrt{2})(1+\sqrt{2}-\sqrt{2})(1+\sqrt{2}-2)(1+\sqrt{2}-\sqrt{2})}$ Sq. unit

$= \sqrt{(\sqrt{2}+1)\cdot 1(\sqrt{2}-1)\cdot 1}$ Sq. unit

$= \sqrt{(\sqrt{2})^2 - 1^2}$ Sq. unit

$= \sqrt{2-1}$ Sq. unit

$= 1$ Sq. unit

$\therefore$ The area of of the quadrilateral *ABCD* = 2 × 1 sq. unit

= 2 sq. unit.

Remarks : By squaring the length of a square, the area is also to be found. By multiplying the length and the breath of a rectangle, the area is also to be found. But the area of a quadrilateral is not to be determined.

Example 2: Draw the quadrilateral having its vertices at the points *A*(-1, 1), *B*(2, -1), *C*(3, 3) and *D*(1, 6), find the length of its each side and one of its diagonals. Hence find the area of the quadrilateral.

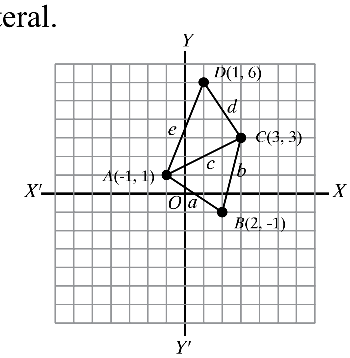


Fig : 11.14

Solution : By plotting the points the quadrilateral $ABCD$ on the plane xy has been shown. Of the quadrilateral $ABCD$,

We have, Side, $AB = a = \sqrt{3^2+2^2} = \sqrt{13}$ unit

Side, $BC = b = \sqrt{1^2+4^2} = \sqrt{17}$ unit

Side, $CD = d = \sqrt{2^2+3^2} = \sqrt{13}$ unit

Side, $DA = e = \sqrt{2^2+5^2} = \sqrt{29}$ unit

Diagonal= $AC = c = \sqrt{(3+1)^2+(3-1)^2}$ unit

$= \sqrt{4^2+2^2} = \sqrt{20}$ unit

In the triangle ΔABC, $2s = a+b+c = \sqrt{13}+\sqrt{17}+\sqrt{20}$ unit

$= 12\cdot 2008 = 3\cdot 6056 + 4\cdot 1231 + 4\cdot 472$ unit

$\Rightarrow s = 6\cdot 1004$ unit

The area of the triangle $ABC = \sqrt{s(s-a)(s-b)(s-c)}$ sq. unit

$= \sqrt{6\cdot 1004 \times 2\cdot 4948 \times 1\cdot 9773 \times 1\cdot 6283}$ sq. unit

$= \sqrt{49\cdot 000}$ sq. unit

$= 7$ sq. unit

In ΔACD, $2s = c+d+e = \sqrt{20}+\sqrt{13}+\sqrt{29}$ unit

$= 4\cdot 4721 + 3\cdot 6056 + 5\cdot 3852$ unit

$= 13\cdot 4429$ unit

$\therefore\ s = 6\cdot 7312$ unit

$\therefore$ Area of $\Delta ACD = \sqrt{s(s-e)(s-d)(s-c)}$ sq. unit

$= \sqrt{6\cdot 7312 \times 2\cdot 2591 \times 3\cdot 1256 \times 1\cdot 3460}$ sq. unit

$= \sqrt{63\cdot 9744}$ sq. unit

$= 7\cdot 9983$ sq. unit

$\therefore$ Area of the quadrilateral $ABCD = (7\cdot 000 + 7\cdot 998)$ sq. unit

$= 14\cdot 998$ sq. unit

= 15 sq. unit (nearly).

Remarks: The quadrilateral $ABCD$ is not a square or a rectangle or a parallelogram or a rhombus. This method is very useful in determining the obtuse shaped land.

Example 3. The coordinates of the four points are respectively $A(2,-3)$, $B(3,0)$, $C(0,1)$ and $D(-1,-2)$.

(a) Show that $ABCD$ is a rhombus.

(b) Find the length of AC and BD and ascertain whether $ABCD$ is a square

(c) Find the area of the quadrilateral by the region of triangle.

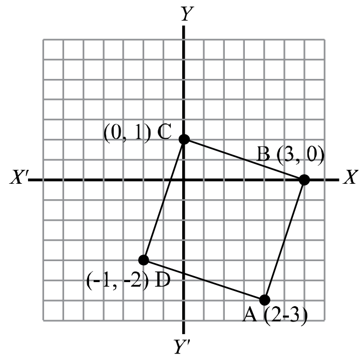


Fig : 11.15

Solution : By plotting the points, the quadrilateral *ABCD* has been shown in the figure 11.15.

(a) Let *a,b,c,d* be the length of the sides *AB, BC, CD* and *DA* respectively and the diagonal $AC = e$ and the diagonal $BD = f$.

Then, $a = \sqrt{(3-2)^2 + (0+3)^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$ unit

$b = \sqrt{(0-3)^2 + (1-0)^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$ unit

$c = \sqrt{(-1-0)^2 + (-2-1)^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$ unit

$d = \sqrt{(2+1)^2 + (-3+2)^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$ unit

Since a = b = c= d = $\sqrt{10}$ unit

$\therefore$ $ABCD$ is a rhombus.

Diagonal $AC = e = \sqrt{(0-2)^2 + (1+3)^2} = \sqrt{4+16} = \sqrt{20}$ unit

And diagonal $BD = f = \sqrt{(-1-3)^2 + (-2-0)^2} = \sqrt{4^2 + 2^2} = \sqrt{20}$ unit

$\therefore$ It is seen $AC = BD$ i.e., the two diagonals are equal.

$$AC^2 = \left(\sqrt{20}\right)^2 = 20$$

$$AB^2 + BC^2 = \left(\sqrt{10}\right)^2 + \left(\sqrt{20}\right)^2 = 10 + 10 = 20$$

$$AC^2 = AB^2 + BD^2.$$

$\therefore$ According to the theorem of Pythagoras, $\angle ABC$ is a right angle.

$\therefore$ The quadrilateral is a square.

$\therefore$ $ABCD$ is a square.

Area of the quadrilateral $ABCD$ = 2 × area of the triangle ΔABC

Here in the case of ABC

$$s = \frac{a+b+c}{2} = \frac{\sqrt{10} + \sqrt{10} + \sqrt{20}}{2}$$

$= \frac{2\sqrt{10}+2\sqrt{5}}{2} = \sqrt{10}+\sqrt{5}$ unit

$\therefore$ Area, $\Delta ABC = \sqrt{s(s-a)(s-b)(s-f)}$ Sq. unit

$= \sqrt{(\sqrt{10}+\sqrt{5})(\sqrt{10}+\sqrt{5}-\sqrt{10})(\sqrt{10}+\sqrt{5}-\sqrt{10})(\sqrt{10}+\sqrt{5}-\sqrt{20})}$ Sq. unit

$= \sqrt{(\sqrt{10}+\sqrt{5})\cdot\sqrt{5}\cdot\sqrt{5}(\sqrt{10}-\sqrt{5})}$ Sq. unit

$= \sqrt{5\cdot(\sqrt{10})^2-(\sqrt{5})^2\}} = \sqrt{5}\cdot(10-5)$ Sq. unit

$= \sqrt{5\cdot 5} = 5$ Sq. unit

$\therefore$ Area of the quadrilateral $ABCD = 2 \times 5$ sq. unit = 10 sq. unit.

Remarks: Easy method: the area of the a square $ABCD$ is $(\sqrt{10})^2 = 10$ sq. unit.

Determination of the Area of the Triangle

Method 2: By this method the area of a triangle is to be determined very easily with the help of the coordinates of the three vertices of a triangle. If the coordinates of the vertices of any polygon are known, the area of the polygon is also to be determined in the same way. In the case of real life, it is not possible to use this method. The reason is, if we want to determine the area of a land and if the shape of the land is of triangled or squared, the area will not be determined by this method. Since the coordinates of the anglic points are not known or are not possible to know. But we can easily measure the length of the side and determine the area by the method no. 1. So, it is necessary for the students to have conception about both methods. The tricks of determining the area of the triangles and the polygons with the help of the method no.2 will be discussed with examples.

The general formula of determining the area of the Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the three vertices of the triangle *ABC*. Like the figure similar to 11.15, the points *A, B* and *C* are arranged in anti-clockwise order.

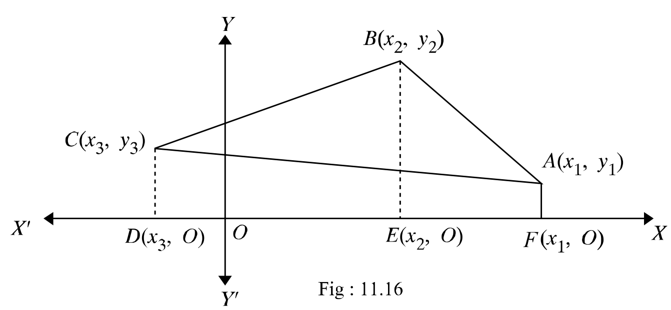


Fig : 11.16

From the figure we get,

Area of polygon $ABCDF$ = area of triangle ABC + area of trapezium $ACDF$ = area of trapezium $ABEF$ +Area of trapezium $BCDF$

Therefore we get,

Area of triangle ABC = area of trapezium $ABEF$ + area of trapezium $BCDE$ – Area of trapezium $ACDF$.

$\therefore$ Area of triangle ABC

$$= \frac{1}{2}\times(BE+AF)\times EF+\frac{1}{2}\times(CD+BE)\times DE-\frac{1}{2}\times(CD+AF)\times DF$$

$$= \frac{1}{2}(y_2+y_1)(x_1-x_2)+\frac{1}{2}(y_3+y_2)(x_2-x_3)-\frac{1}{2}(y_3+y_1)(x_1-x_3)$$

$$= \frac{1}{2}(x_1y_2+x_2y_3+x_3y_1-x_2y_1-x_2y_1-x_3y_2-x_1y_3)$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} \text{ sq. unit}$$

Where,

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} = (x_1y_2+x_2y_3+x_3y_1-x_2y_1-x_2y_1-x_3y_2-x_1y_3)$$

Where in the side of multiplication as the positive sign $\searrow$ it has been obtained $x_1y_2+x_2y_3+x_3y_1$ and as the negative sign $\nearrow$ it has been obtained $-x_2y_1-x_3y_2-x_1y_3$

> Therefore,
>
> $$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

Remark: It is very important to remember that in applying this formula, the vertices $\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$ must be taken in anti-clockwise order.

Example 1. Find the area of the triangle with the vertices $A(2, 3)$, $B(5, 5)$ and $C(-1, 4)$.

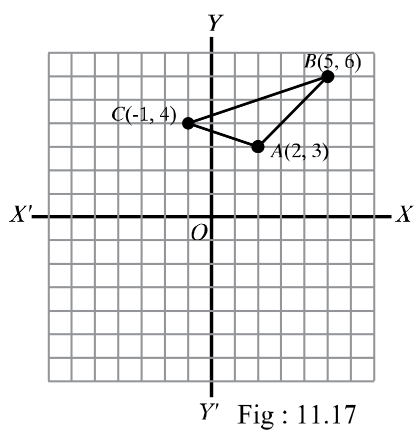


Fig : 11.17

Solution : The vertices $A(2, 3)$, $B(5, 5)$ and $C(-1, 4)$ are taken in anti-clockwise order.

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 2 & 5 & -1 & 2 \\ 3 & 6 & 4 & 3 \end{vmatrix}$ sq. unit

$= \frac{1}{2}(12+20-3-15+6-8)$ sq. unit

$= \frac{1}{2}(12)$ sq. unit

$= 6$ sq. unit.

Example 2. The vertices of a triangle are $A(1,3)$, $B(5,1)$ and $C(3,r)$. If the area of ΔABC is 4 sq. unit, find the possible value of r.

Solution : Considering the vertices $A(1,3)$, $B(5,1)$ and $C(3,r)$ in anticlockwise order, the area of ΔABC

$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 3 & 1 \\ 3 & 1 & r & 3 \end{vmatrix}$ sq. unit

$= \frac{1}{2}(1+5r+9-15-3-6)$ sq. unit

$= \frac{1}{2}(4r-8) = (2r-4)$ sq. unit

According to the question, $(2r-4) = \pm 4$

i.e., $2r = 0$ or, 8

$\therefore$ $r = 0$ or, 4

Answer : $r = 0 \,.\, 4$

Area of Quadrilateral

In the figure 11.17, $ABCD$ is a quadrilateral; its vertices respectively $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$ and A,B,C,D have been arranged in anti-clockwise order.

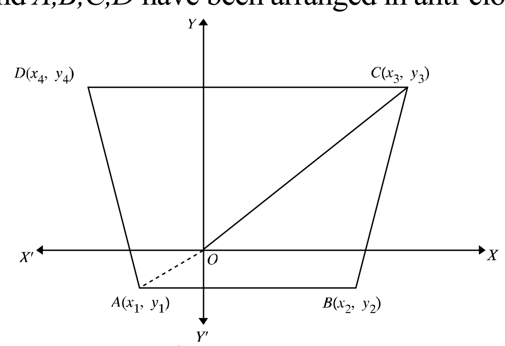


Fig : 11.17

Now, area of the quadrilateral $ABCD$

= Area of the triangle ABC + area of the triangle ACD

$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & x_3 & x_4 & x_1 \\ y_1 & y_3 & y_4 & y_1 \end{vmatrix}$ sq. unit

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1x_3) \text{ sq. unit}$$

$$+ \frac{1}{2}(x_1y_3 + x_3y_4 + x_4y_1 - x_3y_1 - x_4y_3 - x_1y_4) \text{ sq. unit}$$

$$= \frac{1}{2} x_1y_2 + x_2y_3 + x_4y_4 + x_4y_1 - x_2y_1 - x_3y_2 - x_4y_3 - x_1y_4) \text{ sq. unit}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix} \text{ sq. unit}$$

$\therefore$ Area of Quadrilateral $ABCD = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix}$ sq. unit

Similarly, if $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$ and $E(x_5, y_5)$ are the vertices of a pentagon $ABCDE$ (figure 11.18), and if the vertices are arranged in the anticlockwise order, the area of the pentagon $ABCDE$ is the sum of the area of the three triangles ABC, ACD and ADE.

Like the area of the triangle and the quadrilateral, just in the same way, the area of the pentagon $ABCDE$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_1 \end{vmatrix}$$

If the coordinates of the vertices of any polygon are to be known, in the same way we can determine the area easily following the above method.

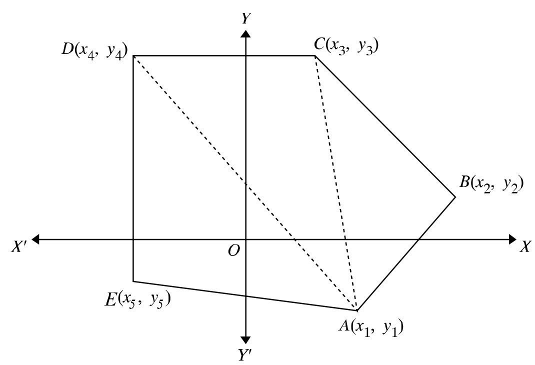


Fig : 11. 18

Activity : Establish the formula for the area of pentagon and hexagon with the help of the method of determining the area of quadrilateral.

Example 3. Find the area of the quadrilateral $ABCD$ with vertices $A(1, 4)$, $B(-4, 3)$, $C(1, -2)$ and $D(4, 0)$.

Solution : Taking the vertices in anti-clockwise order, the area of the quadrilateral $ABCD$

$$= \frac{1}{2}\begin{vmatrix} 1 & -4 & 1 & 4 & 1 \\ 4 & 3 & -2 & 0 & 4 \end{vmatrix} \text{ sq. unit}$$

$$= \frac{1}{2}(3 + 8 + 0 + 16 + 16 - 3 + 8 - 0) \text{ sq. unit}$$

$$= \frac{1}{2}(48) = 24) \text{ sq. unit}$$

Exercise 11·2

1. $A(-2,0),\ B(5,01),\ C(1,4)$ are respectively the vertices of ΔABC.
   (i) Find the lengths of the sides $AB,\ BC$ and CA and the perimeter of ΔABC.
   (ii Find the area of the triangles.
2. In each case find the area of the triangle ABC:
   (i) $A(2,3),\ B(5,6)$ and $C(-1,4)$;
   (ii) $A(5,2),\ B(1,6)$ and $C(-2,-3)$;
3. Show that the points, $A(1,1),\ B(4,4),\ C(4,8)$ and $D(1,5)$ are the vertices of a parallelogram. Find the lengths of the sides AC and BD. Find the area of the parallelogram upto three places of decimals.
4. What is the area of the quadrilateral $ABCD$, with the verices $A(-a,0),\ B(0,-a)$,
   $C(a,0)$ and $D(0,a)$?
5. Show that the four points $(0,-1),\ (-2,3),\ (6,7)$ and $(8,3)$ are the vertices of a rectangle. Find the lengths of its diagonals and the area of the rectangle.
6. If $AB = BC$ holds for the coordinates of the three points respectively $A(-2,1),\ B(10,6)$ and $C(a,-6)$, find the possible value of a. Then find the area of the triangle formed with the help of the value of 'a'.
7. The coordinates of the the three points *A,B,C* are respectively $A(a,a+1),\ B(-6,-3)$ and $C(5,-1)$. If the length of *AB* is twice of *AC*, find the possible value of 'a' and the properties of the triangle.
8. Find the area of the quadrilateral as follows. [Use the method 2]:
   (i) $(0,0),(-2,4),(6,4),(4,1)$; (ii)
   $(1,4),(-4,3),(1,-2),(4,0)$;
   (iii) $(1,0),(-3,-3),(4,3),(4,1)$;
9. Show that the polygon with vertices, $A(2,-3), B(3,-1), C(2,0), D(-1,1)$ and $E(-2,-1)$ has area 11 sq. units.
10. The vertices of a quadrilateral, arranged in anti-clockwise order, are $A(3,4), B(-4,2), C(6,-1)$ and $D(p,3)$. Find the value of P if the area of the quadrilateral *ABCD* is twice the area of the triangle ABC.

11·4 Gradient or Slope of a Straight Line

In this part of Coordinate Geometry, at first we shall discuss what *Gradient* or *Slope* means and how to determine *the Slope* or *the Gradient* of a straight line. By using the concept of the slope how the algebraic form of the straight line appears to be will be discussed. If any straight line passes through two points, the nature of that straight line and the determination of the equation of that straight line are mainly the subject-matter of the discussion. If the two staight lines meet or intersect at any point, by determining the cordinates of that intersceting point and by denoting the straight line with the three equations, the formed triangle will be discussed.

Here we will show that in algebra the single equation of two variables denotes a straight line and their solution is that intersecting point.

Gradient or Slope

In the figure 11.19, we consider the straight line *AB*. The line passes through the two points *A*(2, 3) and *B*(6, 7). According to the figure, the line produces an angle θ with the positive side of x-axis. The angle θ is the measurement of the inclination of the straight line *AB* with the horizontal *x-axis*.

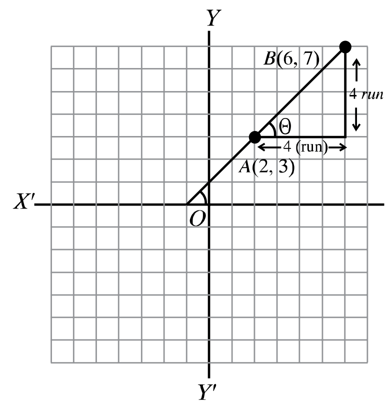


Fig : 11.19

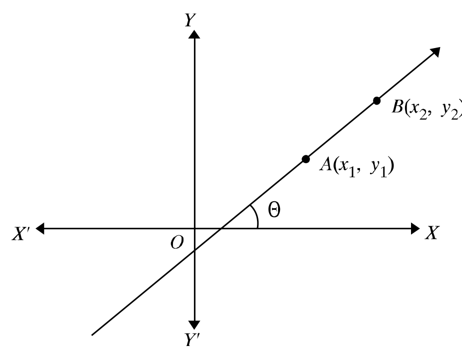
In Coordinate Geometry, we measure the Gradiant *m* of the line *AB* in the following way-

$$m\,M = \frac{\text{Change of the coordinate of } y}{\text{Change of the coordinate of } x} = \frac{7-3}{6-2} = \frac{4}{4} = 1$$

$\therefore$ The gradiant of the line *AB (m)* = 1

Generally, when a straight line *AB* passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ we expresses the gradient *(m)*

$$m = \frac{y_2 - y_2}{x_1 - x_1} \quad \left[\frac{rise}{run}\right]$$



In reality, the relation between the slope *m* and the angle θ produced by any straight line with the positive side of *x-axis* is, $m = \tan\theta$

In figure 11.19, in case of the line *AB* the slope of the straight line is, *m* = 1

i.e., $\tan\theta = 1$

or, $\theta = 45$ (an acute angle).

Example 1. In each of the following cases find the slope of the straight line passing through the given pair of points:

(a) *A*(2,3) and *B*(3,6)

(b) A' ' (2, 1) and B'(- 1, 4)

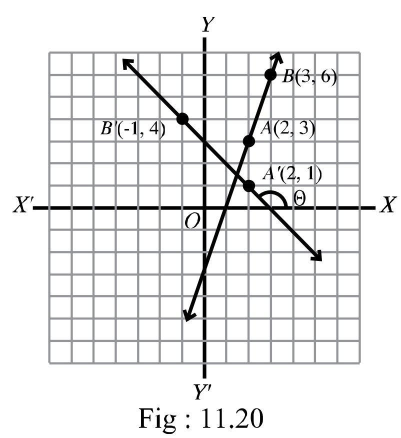


Fig : 11.20

Solution : *(a)* Slope of the line *AB* is = $\frac{rise}{run}$

$= \frac{6-3}{3-2} = \frac{3}{1} = 3$

(b) Slope of the line *AB* is = $\frac{rise}{run} = \frac{4-1}{-1-2} = \frac{3}{-3} = -1$.

Note : In the figure 11.20, it is seen that the slope of the line *AB* is positive and the angle produced is an acute angle. Again in the same figure, it is clear that the slope of the line $A'B'$ is negative and the angle produced is an obtuse angle.

Therefore, from the above discussion we come to the conclusion, if the slope is positive, the angle produced by the line with the positive side of the x-axis is an acute angle and if the slope is negative, the angle produced by the line with the positive side of the x-axis is an obtuse angle.

If the produced angle is zero or right angle, what the slope will be that has been explained with the help of the following example.

Example 2. The coordinates of the three points *A*, *B* and *C* are respectively $(2,2)$, $(5,2)$ and $(2,7)$. Draw the lines *AB* and *AC* on the Cartesian plane. If possible, find the slopes of the lines *AB* and *AC*.

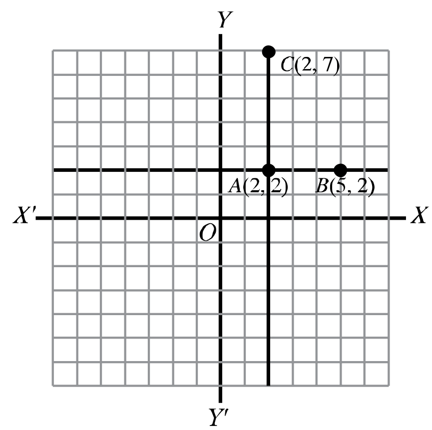


Fig : 11.21

Solution : The lines *AB* and *AC* have been drawn on the Cartesian plane. From the figure, it is observed that the line *AB* is parallel to the *x*-axis, while the line *AC* is parallel to the *y-axis*.

Slope of the line *AB* is

$$m = \frac{rise}{run} = \frac{2-2}{5-2} = \frac{0}{3} = 0.$$

Slope of the line *AC* can not be determined by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$

because $x_1 = x_2 = 2$ and $x_2 - x_1 = 0$

If $x_1 = x_2$, the slope of the line is not determined but the line is parallel to the *y*-axis.

Generally any straight line passes through the point A (x_1, y_1) and B (x_2, y_2)

Slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$ Or, $m = \frac{y_1 - y_2}{x_1 - x_2}$ if $x_1 \neq x_2$

Observe: If $x_1 = x_2$, the line is parallel to the *y-axis* i.e., a parpendicular on the *x-axis*. It is not possible to walk on the perpendicular line or the vertical line. So, the determination of the slope is not possible.

Remarks: In the figure 11.21, on any point of the line *AB,* the ordinate i.e., $y = 2$ and on any point of the line *AC,* the abscissa i.e., $x = 2$. Therefore, the equation of the straight line *AB* is $y = 2$ and the equation of the straight line *AC* is $x = 2$.

Example 3. Find the slope of the line passing through the points $A(-3, 2)$ and $B(3, -2)$.

Solution : If the slope of the line AB is m,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run} = \frac{2-(-2)}{-3-3} = \frac{4}{-6} = -\frac{2}{3}.$$

As the slope is negative, the angle produced by the line with the positive direction of the *x-axis* is an obtuse angle.

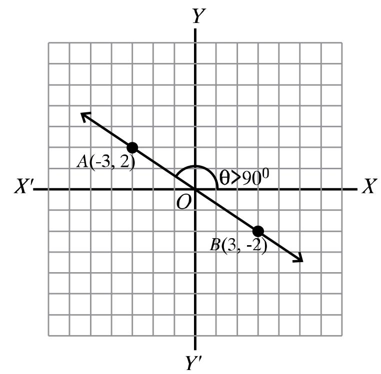


Fig : 11.22

Example 4. What is the value of t if the three points $A(1, -1)$, $B(2, 2)$ and $C(4, t)$ are collinear?

Solution : If the points A, B, C are collinear, the lines AB and BC will have the same slope. Therefore we get,

$$= \frac{2+1}{2-1} = \frac{t-2}{4-2}$$

or, $\frac{3}{1} = \frac{t-2}{2}$

or, $t - 2 = 6$

or, $t = 8.$

So the value of t is 8.

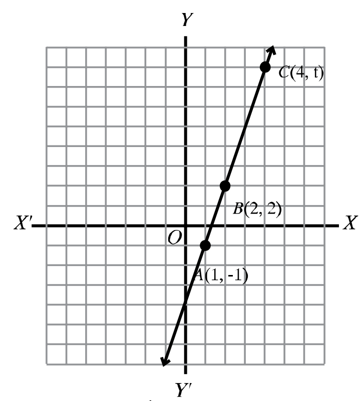


Fig : 11.23

Example 5.

$A(t, 3t)$, $B(t^2, 2t)$, $C(t-2, t)$ and $D(1,1)$

are the four different points. If the lines AB and CD are parallel, find the admissible value of t.

Solution : Slope of the line AB is $m_1 = \frac{2t-3t}{t^2-t} = \frac{-t}{t(t-1)} = \frac{-1}{1-t}$;

Slope of the line CD is $m_2 = \frac{1-t}{1-t+2} = \frac{1-t}{3-t}$.

As the lines AB and CD are parallel, the lines AB and CD have the same slope. i.e. $m_1 = m_2$

or, $\frac{-1}{1-t} = \frac{1-t}{3-t}$.

or, $(1-t)^2 = -(3-t)$

or, $1 - 2t + t^2 = -3 + t$

or, $t^2 - 3t + 4 = 0$

or, $t = -1$ and 2

So, the admissible values of t, -1 and 2.

Exercise 11·3

1. In each case below, find the slope of the straight line passing through the points A and B.
   (a) $A(5,-2)$ and $B(2,1)$
   (b) $A(3,5)$ and $B(-1,-1)$
   (c) $A(t,t)$ and $B(t^2,t)$
   (d) $A(t,t+1)$ and $B(3t,5t+1)$
2. The three different points $A(t,1)$, $B(2,4)$ and $C(1,t)$ are collinear; find the value of t.
3. Show that the points $A(0,-3)$, $B(4,-2)$ and $C(16,1)$ are collinear.
4. If the points $A(1,-1)$, $B(t,2)$ and $C(t^2,t+3)$ are collinear, find the admissible value of t.
5. Find the value of P if the line joining the points $A(3,3p)$ and $B(4,p^2+1)$ has slope – 1.
6. Prove that the points $A(a,0)$, $B(0,b)$ and $C(1,1)$ are collinear if $\frac{1}{a}+\frac{1}{b}=1$.
7. If the points $A(a,b)$, $B(b,a)$ and $C\left(\frac{1}{a},\frac{1}{b}\right)$ are collinear, prove that, $a+b=0$.

11·5 Equation of Straight Lines.

Suppose, a definite straight line $'L'$ passes through two definite points $A(3,4)$ and $B(5,7)$. In the figure 11.24, the line is shown.

Then the slope of the straight line AB is

$$m_1=\frac{7-4}{5-3}=\frac{3}{2} \quad \text{...............(1)}$$

Suppose, $P(x,y)$ is any point on the line L. Then the slope of the line AP is

$$m_2=\frac{y-4}{x-3} \text{...............(2)}$$

Since AB and AP are segments of the same line, both have same slope. i.e.

$$m_1=m_2$$

$$\frac{3}{2}=\frac{y-4}{x-3} \qquad \text{[From (1) and (2), we get]}$$

or, $3x-9=2y-8$

or, $2y-8=3x-9$

or, $2y=3x-1$

or, $y = \frac{3}{2}x - \frac{1}{2}$................(3)

Taking *m* as the slope of the line *PB*

$m_3 = \frac{7-y}{5-x}$...............(4)

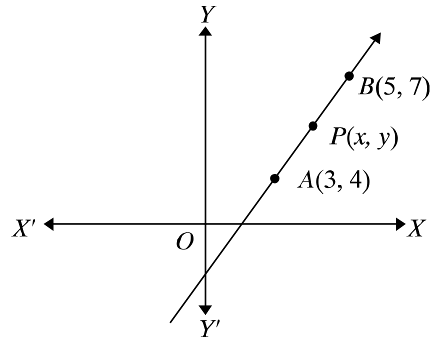


Fig : 11. 24

As the lines *AB* and *PB* have equal slope, [Form (1) and (4) we get]

$m_1 = m_3$

or, $\frac{3}{2} = \frac{7-y}{5-x}$ [From (1) and (4) we get]

or, $15 - 3x = 14 - 2y$

or, $2y + 15 = 3x + 14$

or, $2y = 3x - 1$

or, $y = \frac{3}{2}x - \frac{1}{2}$................(5)

The equation (3) and (5) is the same equation. So, the equation (3) or (5) is the Cartesian equation of the straight line *L*. If we observe, we will find that the equation (3) and (5) is the single equation of *x* and *y* and it indicates a straight line. So, undoubtedly it is said that the single equation of *x* and *y* always indicates a straight line. The equations (3) and (5) is expessed in the following way -

$y = \frac{3}{2}x - 1$ .... (3) or (5)

$\frac{y-4}{x-3} = \frac{3}{2}$ or, $\frac{y-7}{x-5} = \frac{3}{2}$

or, $\frac{y-4}{x-3} = \frac{7-4}{5-3}$ or, $\frac{y-7}{x-5} = \frac{7-4}{5-3}$

or, $\frac{y-4}{x-3} = m$ or, $\frac{y-7}{x-5} = m$

Therefore, it is said usually, if two definite points $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on any straight line, the slope

$m = \frac{y-y_1}{x-x_1} \left[\frac{rise}{run}\right] = \frac{y_2-y_1}{x_2-x_1}$

And the Cartesian equation of that straight line will be –

$\frac{y-y_1}{x-x_1} = m$...................(6)

or, $\frac{y-y_2}{x-x_2} = m$...................(7)

From the equation (6) we get,

$$y - y_1 = m(x - x_1)....................(8)$$

From the equation (7) we get,

$$y - y_2 = m(x - x_2)....................(9)$$

$\therefore$ From (8) and (9) we can say, if the slope of the line is m and the line passes through the definite points $A(x_1, y_1)$ and $B(x_2, y_2)$, the Cartesian equation of the line will be determined by the equation (8) or (9).

From the equation (6) and (7) we get,

$$m = \frac{y - y_1}{x - x_1} = \frac{y - y_2}{x - x_2}..................(10)$$

From the equation (10), it is said clearly, if a straight line passes through two definite points $A(x_1, y_1)$ and $B(x_2, y_2)$, its Cartesian equation will be

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \quad \text{or,} \quad \frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.............(11)$$

Since $m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$

The above discussion is explained with the help of the following examples so that the students can easily understand the slope of the straight line and the equation.

Example 1. Find the equation of the straight line connecting the points $A(3, 4)$ and $B(6, 7)$.

Solution : Slope of the line AB is $m = \frac{rise}{run} = \frac{7-4}{6-3} = \frac{3}{3} = 1$

By applying the equation (8), the equation of the line AB is

$$y - 4 = 1(x - 3)$$

Or, $y - 4 = x - 3$ $\qquad y - y_1 = m(x - x_1)$ ....(8)

Or, $y = x + 1$

By applying the equation (9), the equation of the line AB is

$$y - 7 = 1(x - 6)$$

or, $y = x + 1$ $\qquad y - y_2 = m(x - x_2)$ ....(9)

By applying the equation (11), the equation of the line AB is

$$\frac{y-4}{x-3} = \frac{4-7}{3-6}$$

or, $\frac{y-4}{x-3} = \frac{-3}{-3} = 1$

or, $y - 4 = x - 3$

or, $y = x + 1$

Observed : By applying anyone of the formula (8) or (9) or (11), the equation of the straight line connecting two definite points can be determined. The learners can use anyone according their convenience.

Example 2. The slope of a definite straight line is 3 and the connecting point of the line is $(-2,-3)$. Find the equation of the line

Solution : Given that, slope $m = 3$

The definite point $(x_1, y_1) = (-2,-3)$

$\therefore$ The equation of the line,

$$(y - y_1) = m(x - x_1)$$

or, $y - (-y) = 3(x - (2)]$

or, $y + 3 = 3(x + 2)$

or, $y = 3x + 3$

Example 3. The straight line $y = 3x + 3$ passes through the point $P(t,4)$. Find the coordinate of P. The line intersects the x-axis and the y-axis on the points A and B respectively. Find the coordinates of the points A and B.

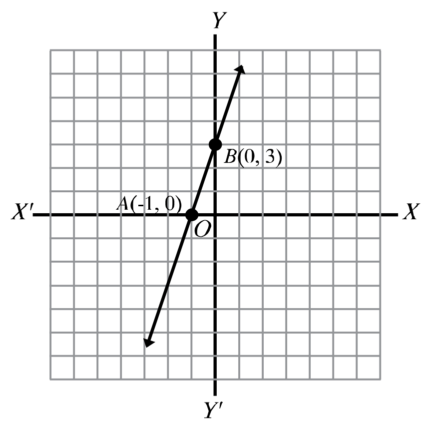


Fig : 11.25

Solution : The point $P(t,4)$ lies on the line $y = 3x + 3$; so coordinates of P will satisfy the equation of the line. i.e.

$$4 = 3.t + 3$$

or, $3t = 4 - 3$

or, $t = \frac{1}{3}$

$\therefore$ The coordinate of P is $P(t,4) = P\left(\frac{1}{3},4\right)$.

The line $y = 3x + 3$ intersects the x-axis at A. So, the ordinate or y coordinate of the point A is 0. [Since y is 0 on all points of the x-axis.]

$\therefore$ $0 = 3x + 3$

or, $x = -1$.

$\therefore$ The coordinate of A is $(-1, 0)$

Again, the line $y = 3x + 3$ intersects the y-axis at B. So, the abscissa or x coordinate of the point B is 0. [Since x is 0 on all points of the y-axis.]

$\therefore$ $y = 3.0 + 3$

or, $y = 3$

$\therefore$ The coordinate of B is $(0, 3)$

Now, draw the line AB on the Cartesian plane.

The line AB intersects the x-axis at the point (-1, 0) and the y-axis at the point (0, 3). i.e., when the value of x is -1, the line $y = 3x + 3$ intersects the x-axis. Again, when the value of y is 3, the line intersects the y-axis. So, the bisector x of the line is -1 and bisector y is 3.

The general equation of such straight line which is Non Vertical can be expressed in the following way.

$$\boxed{y = mx + c}$$

Here, the slope of the line is m and c is the bisector of the y-axis. For $m > 0$ and $c > 0$ of the line is shown in the figure 11.26.

Again, parallel to the y-axis, i.e., the general equation of the perpendicular line on the x-axis is $x = a$. Figure 11.26.

In the same way, parallel to the x-axis, i.e., the general equation of the perpendicular line on the y-axis is $y = b$. Figure 11.26.

Observed, as the value of *'c'* is positive, the line $y = mx + c$ intersects c to the positive side of the y-axis at a single distance. As the value of m $(m = \tan\theta > 0)$ is positive, the angle produced by the line $y = mx + c$ is an acute angle. As the values of $'a'$ I $'b'$ are positive, the line $x = a$ has been on the right side of the y-axis and the line $y = b$ has been shown above the x-axis.

In the case of the negative values of $'a','b'$ I $'c'$, the position of the lines are shown in the figure 11.27.

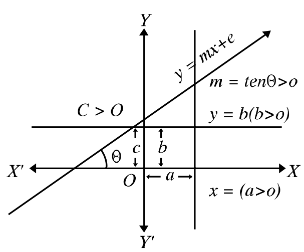


Fig : 11.26

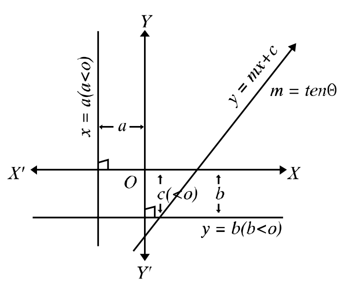


Fig : 11.27

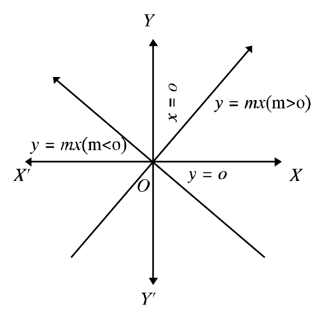


Fig : 11.28

From the figures 11.26 and 11.27 and from above discussion, we can say clearly, if $c = 0$, the line $y = mx$ will pass through the origin $(0, 0)$ and if $a = 0$, the line will pass through y-axis and if $b = 0$, it will pass through x-axis. Figure 11.28

Therefore, the equation of the x-axis is $y = 0$

And the equation of the y-axis is $x = 0$

Example 4. Find the slope and the intersector of the straight line $y - 2x + 3 = 0$. Draw the line on the Cartesian plane.

Solution : $y - 2x + 3 = 0$

or. $y = 2x - 3$ [Shape of $y = mx + c$]

$\therefore$ slope $m = 2$

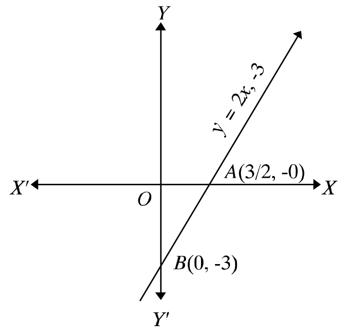


Fig : 11. 29

Intersector of y-axis is $c = -3$.

Now, if the line intersects the x-axis and the y-axis at A and B, we get,

The coordinates of the point $A\left(\frac{3}{2}, 0\right)$ [Putting in x-axis $y = 0$, $x = \frac{3}{2}$]

The coordinates of the point B is $(0, -3)$ [Putting in y-axis $x = 0$, $(y = -3)$]

The line drawn on the Cartesian plane has been shown.

Example 5. The line joining the points $A(-1, 3)$ and $B(5, 15)$ intersects the x-axis and the y-axis at the points P and Q respectively. Find the equation of the line PQ and the length of PQ.

Solution : The equation of the line AB,

$$\frac{y-3}{x+1} = \frac{3-15}{-1-5} = \frac{-12}{-6} = 2$$

or, $y - 3 = 2x + 2$

or, $y = 2x + 5$.............(1)

From (1), the coordinate of P is $\left(-\frac{5}{2}, 0\right)$

and the coordinate of Q is (0, 5)

$\therefore$ the equation of the line PQ,

$$\frac{y-0}{x+\frac{5}{2}} = \frac{0-5}{\frac{-5}{2}-0}$$

Or, $\frac{2y}{2x+5} = \frac{10}{8}$

Or, $2y = 4x + 10$

Or, $y = 2x + 5$

Remarks : AB and PQ are the same straight line.

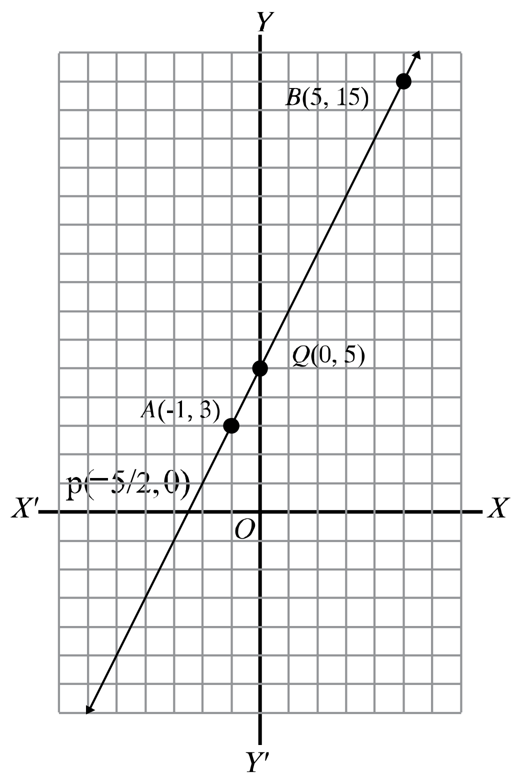


Fig : 11. 30

Now, the length of PQ $= \sqrt{\left(\frac{-5}{2} - 0\right)^2 + (0-5)^2}$

$= \sqrt{\frac{25}{4} + 25} = \sqrt{\frac{125}{4}}$

$= \frac{5\sqrt{5}}{2}$ unit.

Exercise 11·4

1. Observe the following information:

i. In determining the distance between two points we get the help of the theorem of Pythagoras.

ii. The slope of the line y-2x +5 = 0 is -2.

iii. The line 3x +5y =0 is passes through the origin.

Which one of the following is correct?

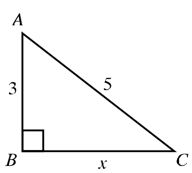
(a) i *(b)* ii and iii *(c)* i and iii *(d)* i, ii and iii

2. In $\{s(s-a)(s-b)(s-c)\}^{\frac{1}{2}}$, *s* means

(a) Area of triangle *(b)* area of circle *(c)* half perimeter of triangle

(d) half perimeter of circle.

3.



Area of the triangle

(a) 12 sq. unit *(b)* 15 sq. unit *(c)* 6 sq. unit *(d)* 60 sq. unit

4.

A(1,1) *B*(3,-3)

Slope of *AB*

(a) 2 *(b)* -2 *(c)* 0 *(d)* 6

5. The product of the two slopes of the two lines x- 2y – 10 = 0 and 2x + y – 3 = 0 is

(a) - 2 *(b)* 2 *(c)* – 2 *(d)* -1

6. The two equations y = $\frac{x}{2}$ + 2 and 2x – 10y +20 = 0

(a) indicate two different lines

(b) indicate the same line

(c) indicate that the two lines are parallel

(d) indicate that the two lines intersect each other.

7. The intersecting point of y = x -3 and y = -x + 3 is
   (a) (0, 0) *(b)* (0, 3) *(c)* (3, 0) *(d)* (-3, 3)
   Answer the questions 8 and 9 according to the information given below
   $x = 1, y = 1$
8. The coordinate of the point on which the two lines intersect x-axis
   (a) (0, 1) *(b)* (1, 0) *(c)* (0, 0) *(d)* (1, 1)
9. The area of the region which is created by the two lines withthe two axes is
   (a) $\frac{1}{2}$ sq. unit *(b)* 1 sq. unit *(c)* 2 sq. unit *(d)* 4 sq. unit
10. Find the equation of the straight line which passes through the point $(2, -1)$ and whose slope is 2.
11. Find the equation of the straight line passing through each pair of points below:
    (a) $A(1, 5),\ B(2, 4)$
    (b) $A(3, 0),\ B(0, -3)$
    (c) $A(a, o),\ B(2a, 3a)$
12. In each case given below, find the equation of the straight line
    (a) Slope is 3 and intersector of y is -5
    (b) Slope is -3 and intersector of y is -5
    (c) Slope is 3 and intersector of y is 5
    (d) Slope is -3 and intersector of y is 5
    Draw these four straight lines on the same plane.
    [By these lines it will be understood in which quadrant the slope and for the symbol indicating bisector will remain]
13. Find the points where each of the following straight lines intersects the *x*-axis and the *y*-axis. Also draw the lines.
    (a) $y = 3x - 3$
    (b) $2y = 5x + 6$
    (c) $3x - 2y - 4 = 0$
14. Find the equation of the straight line passing through the point $(k, 0)$ and having slope k by *k*. Find k if the line passes through the point $(5, 6)$.
15. Find the equation of the straight line passing through the point $(k^2, 2k)$ and having slope $\frac{1}{k}$. If the line passes through the point $(-2, 1)$, find the possible value of *k*.
16. A straight line with slope $\frac{1}{2}$ passes through the point $A(-2, 3)$. If the line passes through the point $(3, k)$, what is the value of k?

17. A line with slope 3 passes through the point $A(-1, 6)$ and intersects x-axis at the point B. Another line passing through the point A intersects x-axis at the point $C(2, 0)$.

    (a) Find the equations of the lines AB and AC.

    (a) Find the area of ΔABC.

18. Show that the two lines $y-2x+4=0$ and $3y=6x+10$ do not intersect each other. By drawing the two lines, explain why the equation have no solution.

19. The three equations $y=x+5$, $y=-x+5$ and $y=2$ indicate the three sides of a triangle. Draw the triangle and find the area.

20. Find the coordinate of the intersecting point of the two lines $y=3x+4$ and $3x+y=10$. Draw the two lines and find the area of the triangle with x-axis.

21. Prove that the three lines $2y-x=2, y+x=7$ and $y=2x-5$ are concurrent. That is, the lines pass through the same point.

22. $y=x+3, y=x-3, y=-x+3$ and $y=-x-3$ indicate the four sides of a quadrilateral. Draw the quadrilateral and determine the area in three different methods.

23. Given that,

    $3x + 2y = 6$

    (a) Determine the points where the given line intersects the two axes

    (b) Determine the magnitude of the intersected part of the two axes and find the area of the triangle produced by the line with the two axes

    (c) Considering the axes and the line the slope, a cubic with 5 sq. unit height has been made on it whose vertex lies over the origin. Find the whole area of the cubic plane and its volume.

24. Given that the slope of the line passing through the points A (1, 4a) and B (5, a^2 - 1) is -1.

    (a) Show that a has two values.

    (b) Suppose, P,Q,R and S are the four points for the two values of a. Find the area of $PQRS$.

    (c) Is the quadrilateral a parallelogram or a rectangle? Explain your opinion with reasons.

Chapter Twelve

Vectors in a Plane

We have learnt scalar quantities and the use of different mathematical operations on these. But many activities of daily life cannot be explained with the concept of scalar quantity. So, we need the conception of vector quantity. We shall discussed vector quantities in this chapter.

At the end of this chapter, the students will be able to

- describe scalar and vector quantities.
- explain scalar and vector quantities with symbols.
- explain equal vector, opposite vector and position vector.
- explain vector addition and rules of vector addition.
- explain the subtraction of vectors.
- explain the scalar multiple of vectors and a unit vector.
- explain the scalar multiple of vectors and the distributive law.
- solve different geometrical problems using vectors.

12.1 Scalar and Vector quantities

Measurement of things is necessary in all spheres of our daily life. Length of a body, measure of time, amount of money, measurement of volume and temperature are denoted respectively by 5 cm., 3 minutes, 12 taka, 5 litres and $6^\circ C$. For these measurements, it is sufficient to state those quantities with their respective units.

Again, if it is stated that a man starting from a point first travels 4 metres and then 5 metres; then to determine his distance from the starting point, it is necessary to know the direction of his motion. It is not possible to determine correctly how far the man has moved from the starting point unless the exact direction of the motion is known.

The quantity which is completely described by its magnitude with unit is a scalar quantity. Each of length, mass, speed, temperature etc. is a ascalar quantity.

The quantity which for its complete description requires magnitude as well as direction is a vector quantity. Each of displacement, velocity, acceleration , weight, force etc. is a vector quantity.

12.2 Geometrical interpretation of aVector: directed line segment

If one end of a straight line is termed as the initial point and the other end as the terminal point then the straight line is called a directed line segment. The directed line segment whose initial point is A and terminal point is B is denoted by AB.

Each directed line segment is a vector quantity whose measurement is the length of the line segment (represented by $|\overrightarrow{AB}|$ or shortly AB) and whose direction is along the line AB straight from A to B.

Conversely, any vector quantity can be expressed by a directed line segment where the length of the line segment is the measurement of the vector quantity and the direction represented from A to B is the direction of the vector.

Hence, vector quantity and directed line segment are the same. Directed line segments are also called Geometric vectors. Our discussion will be limited to the vectors in a plane.

Support Line: Any vector (directed line segment) which is the part of an unending straight line is called the support line or support of the vector.

Usually a vector is represented by a letter e.g. $\vec{AB} = \underline{u}$. But $\vec{AB}$ represents a vector whose initial point is A and the final point is B, $\underline{u}$ does not represent so.

Activities: 1. The school is situated 3 kilometres to the south of your home. What is your velocity if you require 1 hour to go to the school from your home on foot?
2. What is your velocity if you come to home by bicycle in 20 minutes after the school breaks.

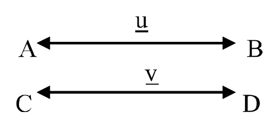
12.3 Equivalence of vectors, Opposite vector

Equal vector: A vector $\underline{u}$ is said to be equal to another vector $\underline{v}$ if

$|\underline{u}| = |\underline{v}|$ [Length of $\underline{u}$ is equal to length of $\underline{v}$]

Supports of $\underline{u}$ and $\underline{v}$ are same or parallel.

Directions of $\underline{u}$ and $\underline{v}$ are same.



It is easily understood that the definition of equivalence abides by the following rules:

$\underline{u} = \underline{v}$

If $\underline{u} = \underline{v}$ then $\underline{v} = \underline{u}$

If $\underline{u} = \underline{v}$ and $\underline{u} = \underline{w}$ then $\underline{v} = \underline{w}$

If the support lines of $\underline{u}$ and $\underline{v}$ are same or parallel then we call briefly $\underline{u}$ and $\underline{v}$ are parallel.

Note: A vector can be drawn at any point which is equal to a given vector. Because if a point P and a vector $\underline{u}$ are given, we draw a straight line at the point P which is parallel to support line of $\underline{u}$. Now we take PQ line segment to the direction of $\underline{u}$ and equal to $|\underline{u}|$. Then according to the construction, $\overrightarrow{PQ} = \underline{u}$

Opposite vector: $\underline{v}$ is called the opposite vector of $\underline{u}$ if

$|\underline{v}| = |\underline{u}|$

Lines of support of $\underline{u}$ and $\underline{v}$ are the same and parallel.

The direction of $\underline{v}$ is opposite of that of $\underline{u}$

If $\underline{v}$ is an opposite vector of $\underline{u}$ then $\underline{u}$ is also opposite of $\underline{v}$. It is clear from the definition of equality that if both $\underline{v}$ then $\underline{w}$ be opposite vectors of $\underline{u}$ then $\underline{v} = \underline{w}$. Therefore, any vector has only one opposite vector.

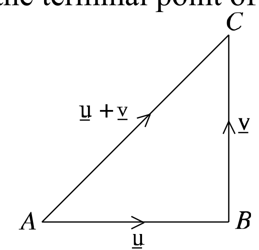
$-\underline{u}$ is the opposite vector of $\underline{u}$.

If $\underline{u} = \vec{AB}$ then $-\underline{u} = \vec{BA}$.

12.4 Addition and Subtraction of vectors

1. (a) Triangle law of addition of vectors

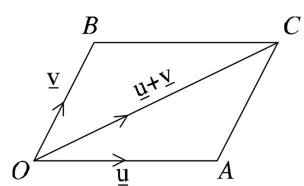
Definition of addition of vectors: If from the terminal point of a vector $\underline{u}$ another vector $\underline{v}$ is drawn, then $\underline{u}+\underline{v}$ denotes such a vector whose initial point is the initial point of $\underline{u}$ and terminal point is the terminal point of $\underline{v}$.



Let $\overrightarrow{AB} = \underline{u}$, $\overrightarrow{BC} = \underline{v}$ be two vectors such that the terminal point of $\underline{u}$ is the initial point of $\underline{v}$ then the vector $\overrightarrow{AC}$ joining the initial point of $\underline{u}$ and the terminal point of $\underline{v}$ is called the sum of the vectors $\underline{u}$ and $\underline{v}$ and is denoted by $\underline{u}+\underline{v}$. If $\underline{u}$ and $\underline{v}$ are not parallel, then the vectors $\underline{u}$, $\underline{v}$ and $\underline{u}+\underline{v}$ form a triangle and hence the above method of addition is known as triangle law.

(b) Parallelogram law of addition of vectors

As a corollary to the triangle law of addition of vectors parallelogram law of addition of vectors is as follows:



If the magnitude and direction of two vectors $\underline{u}$ and $\underline{v}$ are denoted by the two adjacent sides of a parallelogram then the magnitude and direction of $\underline{u}+\underline{v}$ is denoted by that diagonal of the parallelogram which passes through the point of intersection of the lines denoting the two vectors.

Proof: Let OA and OB denote the vectors $\underline{u}$ and $\underline{v}$ drawn from any point O. Draw the parallelogram OACB and its diagonal OC. Then the diagonal OC of the parallelogram will denote the sum of $\underline{u}$ and $\underline{v}$ i.e. $\overrightarrow{OC} = \underline{u}+\underline{v}$ in the parallelogram OACB. We have OB and AC equal and parallel

$\therefore \overrightarrow{AC} = \overrightarrow{OB} = \underline{v}$

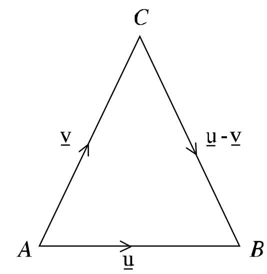
$\therefore\ \underline{u} + \underline{v} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$ [by transfer of vectors].

Note: (1) The sum of two or more vectors is said to be their resultant also. The method of adition of vectors is followed in determining the resultant of forces or velocities.

(2) If the two vectors are parallel, the parallelogram law is not applicable to their addition but the triangle law is applicable in all the cases.

2. Subtraction of vectors

The subtraction of the vectors $\underline{u}$ and $\underline{v}$ is $\underline{u} - \underline{v}$ and it is equivalent to the addition of $\underline{u}$ and $-\underline{v}$ (opposite vector of $\underline{v}$) i.e. $\underline{u} + (-\underline{v})$.



Triangle law of subtraction of vectors

If $\underline{u} = \overrightarrow{AB}$, $\underline{v} = \overrightarrow{AC}$ then $\underline{u} - \underline{v} = \overrightarrow{CB}$ i.e. $\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$

If the initial points of $\underline{u}$ and $\underline{v}$ are same then initial point of $\underline{u} - \underline{v}$ will be the same as the final point of $\underline{v}$ and the final point of $\underline{u} - \underline{v}$ will be the same as the final point of $\underline{u}$.

The difference of two vectors with same initial point is the opposite vector formed by the initial points.

Proof: Line segment CA is produced such that $AE = CA$. $AEFB$ parallelogram is drawn. According to the parallelogram law of additions of vectors $\overrightarrow{AE} + \overrightarrow{AB} = \overrightarrow{AF}$.

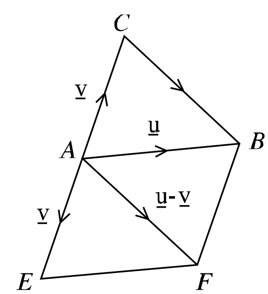
Again AFBC is a parallelogram, as

$BF = AE = CA$ and since BF || AE,then BF || CA.

$\therefore\ \overrightarrow{AF} = \overrightarrow{CB}$ [By transfer of vector]

But $\overrightarrow{AC} = -\underline{v}$ and $AB = \underline{u}$

Therefore, $\underline{u} + (-\underline{v}) = \overrightarrow{CB}$ (Proved)



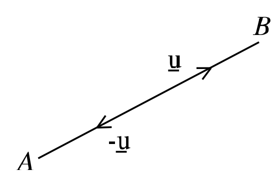
3. Zero Vector

A vector whose absolute value is zero and whose direction cannot be determined is called a zero vector. If $\underline{u}$ is any vector,then what is the value of $\underline{u}$ +(- $\underline{u}$)?

Let $\underline{u} = \overrightarrow{AB}$ then $-\underline{u} = \overrightarrow{BA}$.

$\therefore\ \underline{u} + (-\underline{u}) = \overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA}$ [by triangle law]

But what kind of vector is $\overrightarrow{AA}$? Its initial point and final point are same. Hence its length is zero.



i.e. $\overrightarrow{AA}$ is to be understood as the point A. This kind of vector (whose length is zero) is called zero vector and denoted by the symbol $\underline{0}$. This is the only vector which has no fixed direction and support line.

For the introduction of zero vector we can say that $\underline{u}+(-\underline{u})=\underline{0}$

and $\underline{u}+\underline{0}=\underline{0}+\underline{u}=\underline{u}$

Virtually trial identity is involved with zero vector.

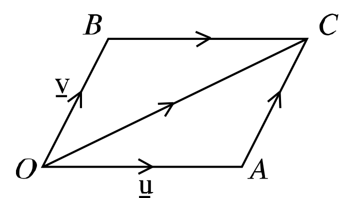
12.5 Laws of Vector Addition

1. Commutative law of addition of Vectors

For any two vectors $\underline{u}$, $\underline{v}$, we get of $\underline{u}+\underline{v}=\underline{v}+\underline{u}$

Proof: Let $\overrightarrow{OA}=\underline{u}$ and $\overrightarrow{OB}=\underline{v}$. Draw the parallelogram OACB and its diagonal OC. OA and BC are equal and parallel .Also OB and AC are equal and parallel.

$\therefore \overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\underline{u}+\underline{v}$



Again, $\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OB}+\overrightarrow{OA}=\underline{v}+\underline{u}$

$\therefore \underline{u}+\underline{v}=\underline{v}+\underline{u}$

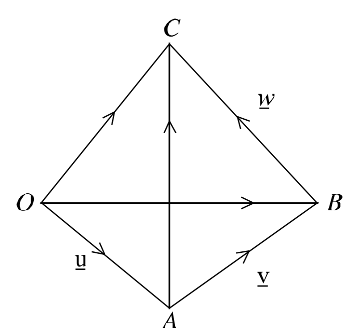
$\therefore$ Addition of Vectors obeys commutative law.

2. Associative law of addition of Vectors

For any three vectors $\underline{u}, \underline{v}, \underline{w}$ we have $(\underline{u}+\underline{v})+\underline{w}=\underline{u}+(\underline{v}+\underline{w})$

Proof: Let $\overrightarrow{OA}=\underline{u}, \overrightarrow{AB}=\underline{v}, \overrightarrow{BC}=\underline{w}$

i.e. $\underline{v}$ is drawn from the terminal of $\underline{u}$ and $\underline{w}$ is drawn from the terminal point of $\underline{v}$ .Join O,C and A,C.



Then $(\underline{u}+\underline{v})+\underline{w}=(\overrightarrow{OA}+\overrightarrow{AB})+\overrightarrow{BC}$

$=\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC}$

Again, $\underline{u}+(\underline{v}+\underline{w})=\overrightarrow{OA}+(\overrightarrow{AB}+\overrightarrow{BC})$

$=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}$

$\therefore (\underline{u}+\underline{v})+\underline{w}=\underline{u}+(\underline{v}+\underline{w})$

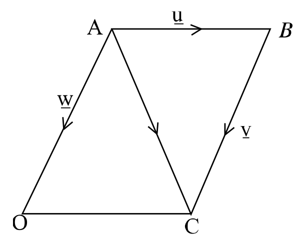
Hence vector addition obeys associative law.

Corollory: The sum of three vectors represented by the three sides of a triangle taken in the same order is zero.

In the diagram above, $\overrightarrow{OB}+\overrightarrow{BA}=\overrightarrow{OA}=(-\overrightarrow{AO})$

$\therefore \overrightarrow{OB}+\overrightarrow{BA}+\overrightarrow{AO}=\overrightarrow{OA}+\overrightarrow{AO}=-\overrightarrow{AO}+\overrightarrow{AO}=\vec{0}$

3. Cancellation Law of addition of vector



For any three vectors $\underline{u}, \underline{v}, \underline{w}$ if $\underline{u}+\underline{v}=\underline{u}+\underline{w}$ then, $\underline{v}=\underline{w}$

Proof: $\underline{u}+\underline{v}=\underline{u}+\underline{w}$

$\therefore \underline{u}+\underline{v}+(-\underline{u})=\underline{u}+\underline{w}+(-\underline{u})$ (adding –u to both sides)

or, $\underline{u}-\underline{u}+\underline{v}=\underline{u}-\underline{u}+\underline{w}$ or, $\underline{v}=\underline{w}$

12.6 Scalar multiple of a vector

If $\underline{u}$ is any vector and m is any real number, then what is understood by $m\underline{u}$ is explained before.

(1) If $m=0$, then $m\underline{u}=0$,

(2) If $m\neq 0$ then the supports of $m\underline{u}$ are same and the length of $m\underline{u}$ is equal to m times that of $\underline{u}$.

(a) If $m>0$, then direction of mu and that of $\underline{u}$ are same.

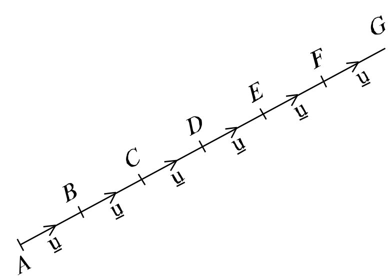
(b) If $m<0$, then direction of mu and that of $\underline{u}$ are opposite.

Note: (1) If $m=0$ or $\underline{u}=\underline{0}$ then $m\underline{u}=\underline{0}$

(2) $1\underline{u}=\underline{u}$, $(-1)\underline{u}=-\underline{u}$

From the above definition, it is observed that, $m(n\underline{u})=n(m\underline{u})=mn(\underline{u})$.

Both $m,n>0$, both <0, any one >0 other < 0, any one or both is zero after considering all these case separately, the reality of the rule can be established.
An Example of such cases is given below



Suppose $\overrightarrow{AB}=\overrightarrow{BC}=\underline{u}$

AC is produced up to G, such that

$CD=DE=EF=FG=AB$

Then $\overrightarrow{AG}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EF}+\overrightarrow{FG}$

$=\underline{u}+\underline{u}+\underline{u}+\underline{u}+\underline{u}+\underline{u}=6\underline{u}$

Again, $\overrightarrow{AG}=\overrightarrow{AC}+\overrightarrow{CE}+\overrightarrow{EG}$

$=2\underline{u}+2\underline{u}+2\underline{u}$

$=3(2\underline{u})$

And $\overrightarrow{AG} = \overrightarrow{AD} + \overrightarrow{DG} = 3\underline{u} + 3\underline{u} = 2(3\underline{u})$

$\therefore 2(3\underline{u}) = 3(2\underline{u}) = 6\underline{u}$

Observation: If support lines of two vectors are alike or parallel, then one can be expressed as scalar multiple of the other. If AB || CD then $\overrightarrow{AB} = m\overrightarrow{CD}$, where $|m| = \frac{|\overrightarrow{AB}|}{|\overrightarrow{CD}|} = \frac{AB}{CD}$.

If $m > 0$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are alike in direction

If $m < 0$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are unlike in direction

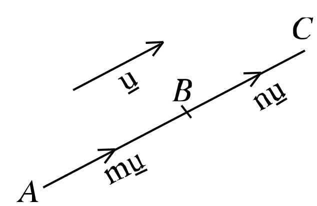
12.7 Distribution laws concerning scalar multiples of vectors

If m, n are two scalars and $\underline{u}$ and $\underline{v}$ are two vectors then

(1) $(m + n)\,\underline{u} = m\underline{u}+n\underline{u}$

(2) $m(\underline{u} + \underline{v})=m\,\underline{u} +m\underline{v}$

Proof: (i) If m or n is zero, then the law is obviously true. Suppose both m and n are positive and $\overrightarrow{AB} = m\underline{u}$ $\therefore |\overrightarrow{AB}| = m\,|\underline{u}|$



AB is produced up to C so that $|\overrightarrow{AB}| = n|\underline{u}|$

$\therefore \overrightarrow{BC} = n\underline{u}$ and

$|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}| = m|\underline{u}| + n|\underline{u}| = (m+n)|\underline{u}|$

$\therefore \overrightarrow{AC} = (m+n)\underline{u}$

But $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ $\therefore m\underline{u} + n\underline{u} = (m+n)\,\underline{u}$

If both m, n are negative, then the length of (m+n)u is $(m+n)|\underline{u}|$ and the direction will be opposite direction of $\underline{u}$, so the length of the vector mu+n$\underline{u}$ will be

$|m||\underline{u}|+|n||\underline{u}|=(|m|+|n|)|\underline{u}|$ [$\because$ the vectors mu, n$\underline{u}$ are in the same direction] and direction will be opposite to the direction of $\underline{u}$.

Because m <0, and n>0, $|m|+|n|=|m+n|$ and in that case also $(m+n)\underline{u}=m\underline{u} +n\underline{u}$ relation is obatinned.

At last if one >0, other<0, then the length of $(m+n)\underline{u}$ will be equal to $|m+n||u|$

(a) alike with the direction of $\underline{u}$ when $|m|>|n|$.

(b) unlike with the direction of $\underline{u}$ when $|m|<|n|$.

Then the length and direction of vector $m\underline{u} + n\underline{u}$ will be alike with $(m+n)\underline{u}$.

Note: Three points A, B, C will be collinear if and only if AC be a scalar multiple of AB.

Comments :

(1)If support lines of two vectors are alike or parallel and their directions are alike, then the vectors are called similar vectors .

(2)The vector whose length is one unit is called a unit vector.

Let $\overrightarrow{OA} = \underline{u}$, $\overrightarrow{AB} = \underline{v}$.

$OB = \overrightarrow{OA} + \overrightarrow{AB} = \underline{u} \; + \underline{v}.$

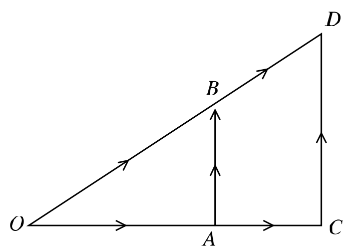


Fig : 1

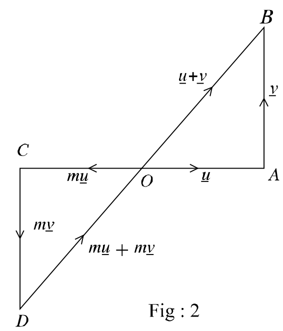


Fig : 2

OA is produced to C . such that OC=m. OA, The straight line CD drawn at C and parallel to AB meets produced OB at D. Since the triangles OAB and OCD are similar. $\dfrac{\left|\overrightarrow{OC}\right|}{\left|\overrightarrow{OA}\right|} = \dfrac{\left|\overrightarrow{CD}\right|}{\left|\overrightarrow{AB}\right|} = \dfrac{\left|\overrightarrow{OD}\right|}{\left|\overrightarrow{OB}\right|} = m$

$\therefore \overrightarrow{CD} = m\overrightarrow{AB} = m\underline{v}$

In figure-1, m is positive and in figure-2 ,m is negative.

$\therefore$ OC = m. OA, CD = m. AB, OD = m.OB

$\overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OD}$ or, $m\left(\overrightarrow{OA}\right) + m\left(\overrightarrow{AB}\right) = m\left(\overrightarrow{OB}\right)$

Note: This formula is true for all values of m.

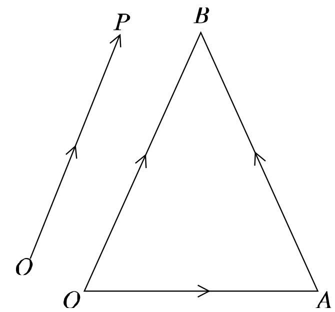
Activities: Verify the formula $(m + n)\,\underline{u} = m\underline{u} + n\underline{u}$ for the vector $\underline{u}$ for different numerical vaues of m and n.

$\therefore m\underline{u} + m\underline{v} = m\left(\underline{u} + \underline{v}\right)$

12.8 Position Vector

With respect to a given point O in a plane, the position of any point P in the plane can be fixed by $\overrightarrow{OP}$. $\overrightarrow{OP}$ is called the position vector of P with respect to O and O is called the vector origin.

Let O be a fixed point in a plane and A is another point in the same plane. The vector $\overrightarrow{OA}$ produced by joining O and A is called the position vector of A with respect to O. Similarly, $\overrightarrow{OB}$ is the position vector of another point B in the same plane with respect to the same point O. Join A, B.



Let $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$

Then, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ i.e. $\underline{a} + \overrightarrow{AB} = \underline{b}$

$\therefore \overrightarrow{AB} = \underline{b} - \underline{a}$

Thus if the position vectors of two points are known, then the vector denoted by the line joining them can be obtained by subtracting the position vector of the initial point from that of the terminal point of the vector.

Note: The position vector of a certain point may be different with respect to different vector origins. In solving a particular problem the position vector of all points of the problem are supposed with respect to the same origin.

Activities: Take a point O as origin on a page of your khatha. Then take another 5 points on it at different positions and draw their position vectors with respect to O.

12.9 Some examples

Example 1. (a) Show that $-(-\underline{a}) = \underline{a}$

(b) $-m(\underline{a}) = m(-\underline{a}) = -m\underline{a}$, where m is a scalar.

(c) $\frac{\underline{a}}{|\underline{a}|}$ is a unit vector when $\underline{a} \neq \underline{0}$

Solution: (a) Using the rule of scalar multiplication of a vector we get,

$$\underline{a} + (-\underline{a}) = \underline{a} + (-1)\underline{a} = \underline{a}(1-1) = 0.\underline{a} = \underline{0}$$

$$\therefore -(-\underline{a}) = \underline{a}$$

(b) $m\underline{a} + (-m)\underline{a} = \{m + (-m)\}\underline{a} = 0.\underline{a} = \underline{0}$

$\therefore (-m)\underline{a} = -m\underline{a} \qquad (1)$

Again $m\underline{a} + m(-\underline{a}) = m\ [\underline{a} + (-\underline{a})] = m\underline{0} = \underline{0}$

$\therefore m(-\underline{a}) = -m\underline{a} \qquad (2)$

From (1) and (2) $(-m)\underline{a} = m(-\underline{a}) = -m\underline{a}$

(c) Let $\hat{\underline{a}}$ unit vector in the direction of $\underline{a}$ and the length of the vector $\underline{a}$ is a i.e. $|\underline{a}| = a$

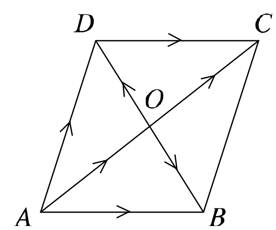
Then $\underline{a} = (a)\ \hat{\underline{a}} = |\underline{a}|\ \hat{\underline{a}}$; but $|\underline{a}|$ is a nonzero scalar because $\underline{a} \neq \underline{0}$.

$\therefore \frac{\underline{a}}{|\underline{a}|} = \frac{|\underline{a}|\hat{\underline{a}}}{|\underline{a}|} = \hat{\underline{a}}$ is a unit vector.

Example 2. ABCD is a parallelogram whose diagonals are AC and BD.

Express the vectors $\overrightarrow{AC}$, $\overrightarrow{BD}$ in terms of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

Express the vectors $\overrightarrow{AB}$, $\overrightarrow{AD}$ in terms of the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$.



Solution : (a) $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AD} + \overrightarrow{AB}$

Again, $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ or, $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$

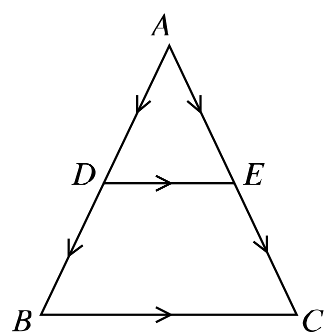
(b) The diagonals of a parallelogram bisect each other.

Now $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \frac{1}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{DB} = \frac{1}{2}\overrightarrow{AC} - \frac{1}{2}\overrightarrow{BD}$

and $\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD} = \frac{1}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{BD}$.

Example 3. With the help of vectors, prove that the line segment joining the middle points of two sides of a triangle is parallel to and half of the third side.

Solution: Let D and E be the middle points of the sides AB and AC of the triangle ABC. D and E are joined. It is required to prove that DE | | BC and $DE = \frac{1}{2}BC$.



By the triangle law of subtraction of vectors,

$\overrightarrow{AE} - \overrightarrow{AD} = \overrightarrow{DE}$................(1)

and $\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$................(2)

But $\overrightarrow{AC} = 2\overrightarrow{AE}$, $\overrightarrow{AB} = 2\overrightarrow{AD}$

[$\because$ D and E are respectively the middle points of AB and AC]

From (2), $\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$

or, $2\overrightarrow{AE} - 2\overrightarrow{AD} = \overrightarrow{BC}$ or, $2\left(\overrightarrow{AE} - \overrightarrow{AD}\right) = \overrightarrow{BC}$

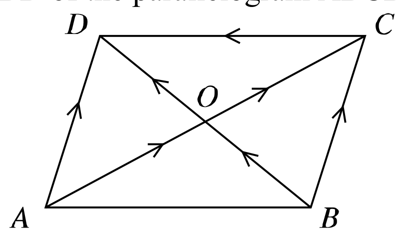
$\therefore 2\overrightarrow{DE} = \overrightarrow{BC}$, [from (1)]

or, $\left|\overrightarrow{DE}\right| = \frac{1}{2}\left|\overrightarrow{BC}\right|$ $\therefore$ $DE = \frac{1}{2}BC$.

Again the lines of support of the vectors $\overrightarrow{DE}$ and $\overrightarrow{BC}$ are same or parallel. But $\overrightarrow{BC}$ and $\overrightarrow{DE}$ cannot be same. Hence the lines of support of the vectors $\overrightarrow{DE}$ and $\overrightarrow{BC}$ i.e. the lines DE and BC are parallel.

Example 4. Prove that by Vector methods that the diagonals of a parallelogram bisect each other.

Let the diagonals AC and BD of the parallelogram ABCD intersect at O.



Suppose, $\overrightarrow{AO} = \underline{a}$, $\overrightarrow{BO} = \underline{b}$, $\overrightarrow{OC} = \underline{c}$, $\overrightarrow{OD} = \underline{d}$.

It is required to prove that, $|\underline{a}| = |\underline{c}|$, $|\underline{b}| = |\underline{d}|$.

Proof : $\overrightarrow{AO} + \overrightarrow{OD} = \overrightarrow{AD}$ and $\overrightarrow{BO} + \overrightarrow{OC} = \overrightarrow{BC}$

Since the opposite sides of a parallelogram are equal and parallel.

$\therefore \overrightarrow{AD} = \overrightarrow{BC}$ i.e., $\overrightarrow{AO} + \overrightarrow{OD} = \overrightarrow{BO} + \overrightarrow{OC}$

or, $\underline{a} + \underline{d} = \underline{b} + \underline{c}$ or, $\underline{a} - \underline{c} = \underline{b} - \underline{d}$ [adding $-\underline{c} - \underline{d}$ to both sides]

But AC is the support line of both $\underline{a}$ and $\underline{c}$; $\therefore$ AC is also the support of $\underline{a} - \underline{c}$.

BD is the support line of both $\underline{b}$ and $\underline{d}$ $\therefore$ BD is also the support of $\underline{b} - \underline{d}$.

If $\underline{a} - \underline{c}$ and $\underline{b} - \underline{d}$ are two equal and nonzero vectors, then their lines of support are same or parallel. But AC and BD are two intersecting straight lines which are not parallel.

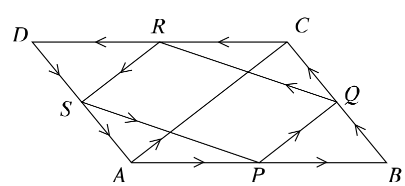
Hence $\underline{a} - \underline{c}$ and $\underline{b} - \underline{d}$ are zero vectors.

$\therefore \underline{a} - \underline{c} = 0$ or, $\underline{a} = \underline{c}$. Also $\underline{b} - \underline{d} = 0$ or, $\underline{b} = \underline{d}$

$\therefore |\underline{a}| = |\underline{c}|$ and $|\underline{b}| = |\underline{d}|$

$\therefore$ The diagonals of a parallelogram bisect each other.

Example 5. Prove by vector method that the straight lines joining the middle points of the adjacent sides of a quadrilateral form of a parallelogram.



Solution: Let P, Q, R, S be the middle points of the sides of the quadrilateral ABCD. Join P and Q, Q and R, R and S, S and P.

It is required to prove that PQRS is a parallelogram.

Proof: Let, $\overrightarrow{AB}=\underline{a}, \overrightarrow{BC}=\underline{b}, \overrightarrow{CD}=\underline{c}, \overrightarrow{DA}=\underline{d}$.

Then, $\overrightarrow{PQ}=\overrightarrow{PB}+\overrightarrow{BQ}=\frac{1}{2}\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}=\frac{1}{2}(\underline{a}+\underline{b})$

Similerly, $\overrightarrow{QR}=\frac{1}{2}(\underline{b}+\underline{c})$, $\overrightarrow{RS}=\frac{1}{2}(\underline{c}+\underline{d})$ and $\overrightarrow{SP}=\frac{1}{2}(\underline{d}+\underline{a})$

But, $(\underline{a}+\underline{b})+(\underline{c}+\underline{d})=\overrightarrow{AC}+\overrightarrow{CA}=\overrightarrow{AC}-\overrightarrow{AC}=0$

i.e. $\underline{a}+\underline{b}=-(\underline{c}+\underline{d})$

Now $\overrightarrow{PQ}=\frac{1}{2}(\underline{a}+\underline{b})=-\frac{1}{2}(\underline{c}+\underline{d})=-\overrightarrow{RS}=\overrightarrow{SR}$

$\therefore PQ$ and SR are equal and parallel. Similarly, it can be proved that QR and PS are equal and parallel. $\therefore$ PQRS is a parallelogram.

Exercise 12

1. If AB | | DC then

   i $\overrightarrow{AB}$ = m. $\overrightarrow{DC}$, where m is a scalar quantity.

   ii $\overrightarrow{AB}=\overrightarrow{DC}$

   iii $\overrightarrow{AB}=\overrightarrow{CD}$

   Which one of the above sentences is true?

   a. i b. ii

   c. i & ii d. i, ii & iii

2. If the two vectors are parallel

   i Parallelogram law is applicable in case of their addition

   ii Triangle law is applicable in case of their addition

   iii Their lengths are always equal

Which one is true among the above sentences?

   a. i b. ii

   c. i & ii d. i, ii & iii

3. Which one of the following is true if AB = CD and AB| | CD?

   a. $\overrightarrow{AB}=\overrightarrow{CD}$ b. $\overrightarrow{AB}$ = m. $\overrightarrow{CD}$ where m>1

   c. $\overrightarrow{AB}+\overrightarrow{DC}$ <O d. $\overrightarrow{AB}$ + m. $\overrightarrow{CD}$ = O where m>1

Answer to the questions 4 & 5 on the basis of the information given below:

C is any point on the line segment AB and $\underline{a}$, $\underline{b}$ & $\underline{c}$ are respectively the position vectors of the points A, B & C with respect to a vector origin.

4. Which one of the following is correct when the point C divides AB internally in the ratio 2 : 3 ?

   a. $\underline{c}=\frac{\underline{a}+2\underline{b}}{5}$ b. $\underline{c}=\frac{2\underline{a}+\underline{b}}{5}$

   c. $\underline{c}=\frac{3\underline{a}+2\underline{b}}{5}$ d. $\underline{c}=\frac{2\underline{a}+3\underline{b}}{5}$

5. Which one of the following is correct if O is the vector origin ?

   a. $\overrightarrow{OA}=\underline{a}-\underline{b}$ b. $\overrightarrow{OA}+\overrightarrow{OC}=\overrightarrow{AC}$

   c. $\overrightarrow{AB}=\underline{b}-\underline{a}$ d. $\overrightarrow{OC}=\underline{c}-\underline{b}$

6. If D, E, F are the middle points respectively of the sides BC, CA, AB of the triangle ABC .

   (a) Express the vectors $\overrightarrow{BC}, \overrightarrow{AD}, \overrightarrow{BE}, \overrightarrow{CF}$ in terms of the vector $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

   (b) Express the vectors $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AD}$ in terms of the vector $\overrightarrow{BE}$ and $\overrightarrow{CF}$.

   (c) Express the vectors $\overrightarrow{AC}, \overrightarrow{BC}, \overrightarrow{AD}$ and $\overrightarrow{CF}$ in terms of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BE}$. Also prove that, $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=0$

7. If $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are the diagonals of the parallelogram ABCD, then express the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of the vectors $\overrightarrow{AD}$ and $\overrightarrow{BD}$ and show that, $\overrightarrow{AC}+\overrightarrow{BD}=2\overrightarrow{BC}$ and $\overrightarrow{AC}-\overrightarrow{BD}=2\overrightarrow{AB}$.

8. Show that: (a) $-(\underline{a}+\underline{b})=-\underline{a}-\underline{b}$

   (b) If $\underline{a}+\underline{b}=\underline{c}$, then $\underline{a}=\underline{c}-\underline{b}$ and conversely.

9. Show that (a) $\underline{a}+\underline{a}=2\underline{a}$ (b) $(m-n)\,\underline{a}=m\underline{a}-n\underline{a}$

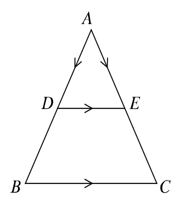
   (c) $m(\underline{a}-\underline{b})=m\underline{a}-m\underline{b}$

10. (a) If each of $\underline{a}$, $\underline{b}$ is a nonzero vector, then $\underline{a}=m\underline{b}$ can be true if and only if $\underline{a}$, is parallel to $\underline{b}$.

    (b) If both $\underline{a}$, $\underline{b}$ are nonzero and non-parallel vectors, and if $m\underline{a}+n\underline{b}=0$ then show that, $m=n=0$

11. If $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ are the position vectors respectively of the points A,B,C,D then show that, ABCD will be a parallelogram if and only if $\underline{b}-\underline{a}=\underline{c}-\underline{d}$.

12. Prove with the help of vectors that the straight line drawn from the middle point of a side of a triangle and parallel to another side passes through the middle point of the third side.
13. If the diagonals of a quadrilateral bisect each other, prove that it is a parallelogram.
14. Prove with the help of vectors that the straight line joining the middle points of the non-parallel sides of a trapezium is parallel to and half of the sum of the parallel sides.
15. Prove with the help of vectors that the straight line joining the middle points of the diagonals of a trapezium is parallel to and half of the difference of the parallel sides.
16.



D and E are respectively the middle points of the sides AB and AC of the triangle ΔABC.

a. Express $(\overrightarrow{AD} + \overrightarrow{DE})$ in terms of the vector $\overrightarrow{AC}$.

b. Prove with the help of vectors that, AB| |DC and DE = $\frac{1}{2}$ BC.

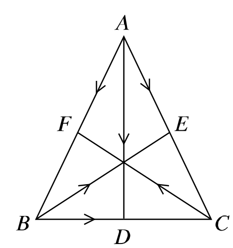
c. If M and N are the middle points of the diagonals of the trapezium ABCD, then prove with the help of vectors that MN| |AD| |BC and MN = $\frac{1}{2}$ (BC-AD).

17. D, E and F are the middle points of the sides BC, CA and AB respectively of the ΔABC.

a. Express $\overrightarrow{AB}$ in terms of the vectors $\overrightarrow{BE}$ and $\overrightarrow{CF}$.

b. Prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{O}$.

c. Prove with the help of vectors that the straight line drawn through F parallel to BC must go through E.



Chapter Thirteen

Solid Geometry

Solids of different shapes are always needed and used in our practical life. Among these, there are regular and irregular solids. The method of determining volumes and areas of surfaces of regular solids and compound solids constructed of two regular solids will be discussed in this chapter.

At the end of this chapter, the students will be able to

- draw the symbolic diagram of a solid
- determine volumes and areas of surfaces of prism, solids of pyramid shape, spheres and right circular cones
- solve problems using the knowledge of solid geometry
- measure volumes and areas of surfaces of compound solids
- apply the knowledge of solid geometry in practical areas.

13.1 Basic concepts

Basic concept of point, line and plane has been discussed in secondary general geometry. In solid geometry point, line and plane are considered as basic concepts.

1. Each of length, breadth and height of a body is called a dimension of the body.
2. A point has no length, breadth or thickness. It is an assumption. For practical purpose, we use a dot (.) for a point. It can be called a replica of position. Hence a point has no dimension. So it is zero-dimensional.
3. A line has length only, but no breadth and height. Hence a line is one-dimensional.
4. A surface has length and breadth, but no height. Hence a surface is two-dimensional.
5. A body having length, breadth and thickness is called a solid. Hence a solid is three dimensional.

13.2 Some elementary definition

1. Plane surface: If the straight line joining any two points on a surface lies wholly in that surface, then the surface is called a plane surface or simply a plane. The upper surface of the still water of a pond is a plane. The smooth floor of a room polished with cement or mosaic is considered to be a plane. But geometrically it is not a plane, for there are high and low points on the floor.

Note:

Unless otherwise mentioned, lines and planes in solid geometry are regarded as infinitely extended. Hence it may be inferred from the definition of a plane that if one part of a straight line lies in a plane then the other part cannot be outside it.

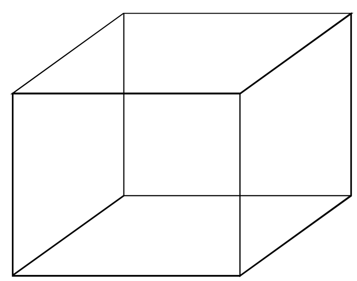
2. Curved surface: If the straight line joining any two points on a surface does not lie wholly in the surface, then the surface is called a curved surface. The surface of a sphere is a crved surface.

3. Solid geometry: The branch of mathematics which concerns with the properties of solids and surfaces, lines and points is called solid geometry.

Sometimes it is called geometry of space or geometry of three dimensions.

4. Coplanar straight line: If two or more straight lines lie in the same plane or a plane can be made to pass through them, then these straight lines are said to be coplanar.

5. Skew or non-coplanar straight line: Straight lines which do not lie in one plane or through which a plane cannot be made to pass are called skew or non-coplanar straight lines. If two pencils are tied cross-wise like a plus sign or a multiplication sign, two non-coplanar lines are formed.



6. Parallel straight lines: Two coplanar straight lines are said to be parallel, when they do not intersect each other, i.e., they have no common point.

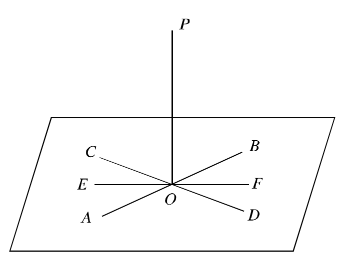
7. Parallel planes or surfaces: Two planes are said to be parallel when they do not intersect, that is, they do not have any common point.

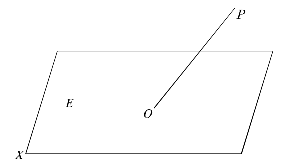
8. Line parallel to a plane: If a plane and a straight li intersect though they are extended indefinitely, then the straight line is said to be parallel to the plane.

Note:

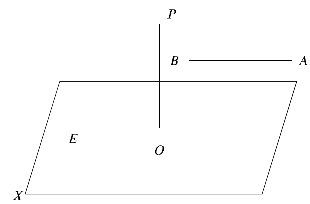
In general, it is to some extent complicated to draw the diagram of a three dimensional body on a two dimensional paper or board. So it would be easier for the students to understand and remember if during class room teaching every definition is explained by drawing a diagram.

9. Normal or perpendicular to a plane: A straight line is said to be normal or perpendicular to a plane when it is perpendicular to every straight line in the plane meets it.





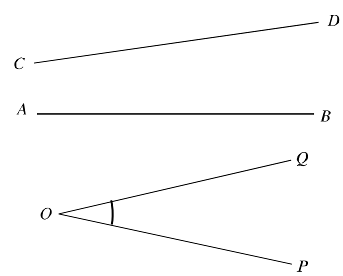
10. Oblique line: A straight line is said to be an oblique line to a plane if it is neither parallel nor perpendicular to the plane.



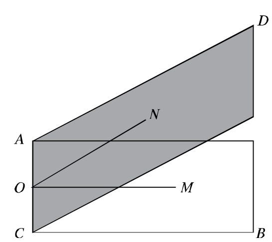
11. Vertical line or plane: A straight line or a plane is said to be vertical when it is parallel to plumb line hanging freely at rest.

12. Horizontal line or plane: A plane is said to be horizontal when it is perpendicular to a vertical line. Again a straight line is said to be horizontal when it is perpendicular to a vertical line or when it lies in a horizontal plane.

13. Plane and skew quadrilateral: A quadrilateral is said to be plane when its sides lie in the same plane. Again a quadrilateral whose sides do not lie in the same plane is called skew quadrilateral. Two adjacent sides of a skew quadrilateral lie in one plane and the other two adjacent sides lie in another plane. Hence the opposite sides of a skew quadrilateral are also skew.

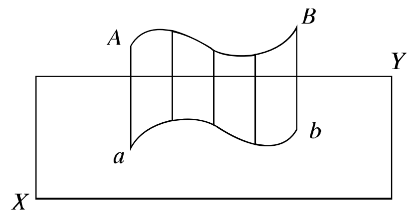


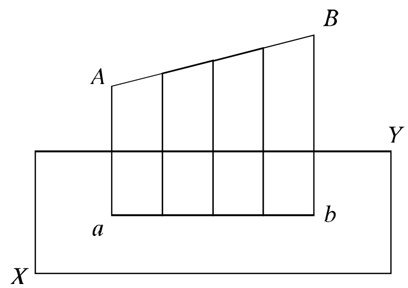
14. Angle between two skew straight lines: The angle between two skew straight lines is the angle between one of them and the line drawn through any point in that line parallel to other. Again if two straight lines parallel to skew straight lines are drawn at a point, then the angle formed at that point is equal to the angle between the skew straight lines. Let AB and CD be the skew lines. Take any point O and through O, draw OP, OQ, parallel to AB, CD respectively. Then the angle POQ indicates the angle between the skew lines AB and CD.



15. Dihedral angle: If two planes intersect in a straight line, then the angle between the two straight lines drawn from any point on the line of intersection and at right angles to it. One in each plane is the dihedral angle between the given planes. The two planes AB and CD intersect along the straight line AC. From O, any point on AC, two straight lines OM in the plane AB and ON in the plane CD are drawn such that each is perpendicular to AC at O. Then <MON is the dihedral angle between the plane AB and CD. Two intersecting planes are said to be perpendicular to each other when the dihedral angle between them is a right angle.

16. Projection: The projection of a point on a given line or a plane is the foot of the perpendicular drawn from the point to the line or plane. The projection of a line straight or curved on a plane is the locus of the feet of the perpendiculars drawn from all points in the given line to the given plane. It is also called orthogonal projection.





In the above diagram the projections of a curved line and a straight line are shown.

13.3 Relation between two straight lines

(a) Two straight lines may be coplanar in which case they must either be parallel or meet in a point.

(b) Two straight lines may be skew in which case they will neither be parallel nor will they meet in a point.

13.4 Axioms

(a) A straight line joining any two points in a plane lies wholly in that plane, though produced indefinitely. Hence if a straight line and a plane have two common points, they will have innumberable common points along the straight line.

(b) An infinite number of planes can be drawn through one or two given straight lines.

13.5 Relation between a straight line and a plane

(a) If a straight is parallel to a plane, then there will be no common point between them.

(b) If a straight line cuts a plane, then they will have one and only one point common to them.

(c) If a straight line and a plane have two common points then the line will completely coincide with the plane.

13.6 Relation between two planes

(a) If two planes are parallel then they will have no common point.

(b) If two planes intersect each other, then they will intersect one another in a straight line and they will have innumerable common points.

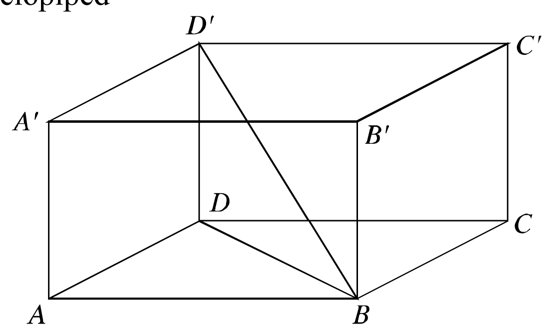
13.7 Solid

We know that a book, a brick, a box or a spherical ball is a solid and occupies some space. Again, a piece of stone or wood, a part of a brick, a fragment of coal, a lump of dried sticky soil etc are also examples of solids. But these are irregular solids. The body enclosed by plane or curved surfaces and occupying some space is called a solid. At least three straight lines are required to enclose a portion of a plane, so also four planes are required to enclose some space. These planes are the faces or the surfaces of the solid and the line in which two such planes intersect is an edge of the solid. A book or a brick has six faces and twelve edges. A cricket ball is enclosed by a curved surface.

Activities: 1. Write the name of a regular and an irregular solid individually. 2. Mention some uses of the solids cited by you.

13.8 Volume and area of surface of uniform solids

1. Rectangular Parallelopiped



Figure

The solid enclosed by three pairs of parallel planes is called a parallelopiped. Each of the six planes is a parallelogram and the opposite faces are congruent. A parallelopiped has twelve edges divided into three groups.

The parallelopiped of which the faces are rectangles is called a rectangular parallelopiped. The rectangular parallelopiped of which the faces are squares is called a cube.

The faces of the rectangular parallelopiped and the faces of the cube in the above diagram are ABCD, A′B′C′D′, BCC′B′, ADD′A′, ABB′A′, DCC′D′ and the edges are AB, A′B′, CD, C′D′, BC, B′C′, AD, A′D′, AA′, BB′, CC′, DD′ and the diagonal is BD′.

Let the length, breadth and height of the rectangular parallelopiped be respectively AB = a units, AD = b units and AA′ = c units .

(a) Area of the whole surface of the rectangular parallelopiped

= Sum of the areas of the six faces

= 2(the area of the face ABCD + the area of the face ABB′A′ + the area of the face ADD′A′)

= $2(ab + ac + bc)$ square units

= $2(ab + bc + ca)$ square units.

(b) Volume = AB × AD × AA′ cubic units = *abc* cubic units

(c) Diagonal $BD' = \sqrt{BD^2 + DD'^2} = \sqrt{AB^2 + AD^2 + DD'^2} = \sqrt{a^2 + b^2 + c^2}$ units

2. For a cube, **a = b = c.** Therefore

(a) Area of the whole surface = $2(a^2+a^2+a^2) = 6a^2$ square units

(b) Volume = a. a. a = a^3 cubic units

(c) Diagonal = $\sqrt{a^2 + a^2 + a^2} = \sqrt{3}\,a$ units

Example 1. A rectangular parallelopiped has its length, breadth and height in the ratio 4:3:2 and the area of its whole surface is 468 square metres; find the diagonal and the volume of it.

Soution: Let the length, breadth and height be respectively $4x, 3x, 2x$ metres.

Then, $2(4x.3x + 3x.2x + 2x.4x) = 468$

or, $52x^2 = 468$ or, $x^2 = 9 \therefore x = 3$

∴ The length of the solid is 12 metres, the breadth is 9 metres and the height is 6 metres.

Hence, the length of the diagonal = $\sqrt{12^2 + 9^2 + 6^2} = \sqrt{144 + 81 + 36} = \sqrt{261}$ metres.

=16.16 metres (approx.)

And volume = 12×9×6 = 648 cubic metres.

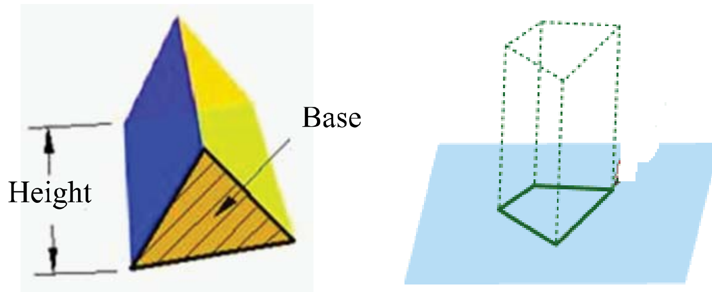
Activities:

1. Measure the length, breadth and height of a hardboard box (cartoon or box containing a bottle of medicine) and find its volume, area of six surfaces and the length of a diagonal.

2. Prism

A prism is a polyhedron, bounded by two parallel polygonal faces and the other faces always being parallelograms. The parallel sides are known as bases and the sides are known as lateral faces. If all the lateral surfaces are rectangular it is called a right prism; otherwise they are called obliques prism. Practically right prisms are frequently used. The prism is named by the shape of its base. For example, triangular prism, quadrilateral prism, pentagonal prism etc.

If the base is a regular polygon, the prism is called a regular prism. If the base is not a regular polygon, the prism is known as an irregular prism. So by definition all rectangular solids and cubes are prisms. Right triangular prism made of glass is used for scattering of light.



Two types of prism

a) The area of total surfaces of a prism

= 2 (area of the base) + area of the lateral surfaces

= 2 (area of the base) + perimeter of the base × height

b) Volume = area of the base × height

Example 2. The lengths of the sides of the base of a triangular prism are 3 cm, 4 cm and 5 cm respectively and height is 8 cm. Find the volume and area of its total surfaces.

Solution: The lengths of the sides of the base of the prism are 3 cm, 4 cm and 5 cm respectively. Since $3^2 + 4^2 = 5^2$, its base is a right angled triangle whose area $= \frac{1}{2} \times 4 \times 3 = 6$ sq. cm.

$\therefore$ The area of all the surfaces $= 2 \times 6 + \frac{1}{2}(3+4+5) \times 8 = 12 + 48 = 60$ sq. cm.

Volume of the prism $= 6 \times 8 = 48$ cubic cm.

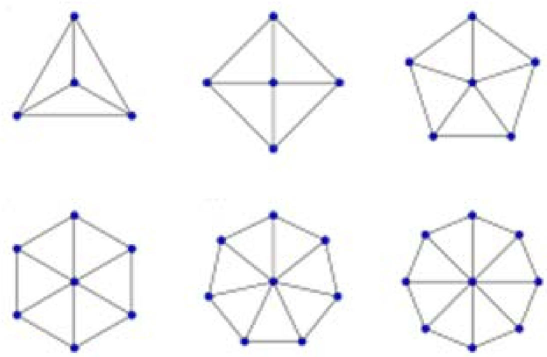
So, the area of all the surfaces is 60 sq. cm. and the volume is 48 cubic cm.

3. Pyramid

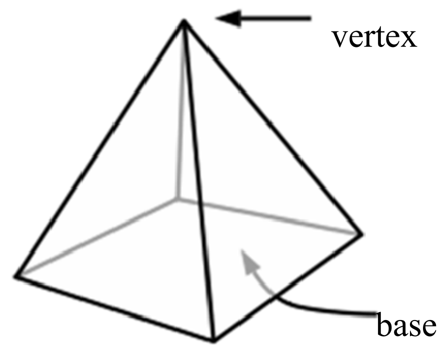
A solid figure with a polygonal base and triangular faces that meet at a common point. The base of a pyramid is a any polygon and its lateral surfaces are of any triangular shape. But if the base is a regular polygon and the lateral faces are congruent triangles, the pyramid is called regular pyramid. The regular pyramids are eye-catching. The line joining the vertex and any corner of the base is called the edge of the pyramid. The length of the perpendicular from the vertex to the base is called the height of the pyramid.

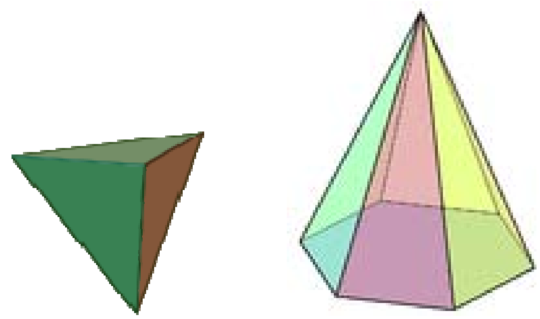
Usually, a solid with a square base and four congruent triangles meeting at a point is considered as a pyramid. These pyramids are in wide use.

A solid enclosed by four equilateral triangles is known as regular tetrahedron which is also a pyramid. This pyramid has 3 + 3 = 6 edges and 4 vertices. The perpendicular from the vertex falls on the centroid of the base.



Bases of different types of pyramids





Pyramids

a) The area of all surafes

= Area of the base + area of the lateral surfaces

But if the lateral surfaces are congruent triangles

The area of all surafes of the pyramid

= Area of the base $+\frac{1}{2}$ (perimeter of the base $\times$ slant height)

If the height of the perimid is h, radius of the inscribed circle of the base is r and l is its slant height, then $l=\sqrt{h^2+r^2}$

b) Volume $=\frac{1}{3}\times$ area of the base $\times$ height

Example 3. The height of a pyramid with a square base of side 10 cm. is 12 cm. Find its area of all surfaces and the volume.

Solution : The perpendicular distance of any side of the base from the centre $r=\frac{10}{2}$ cm. = 5 cm. The hright of the pyramid is 12 cm. Therefore, the slant height of any lateral surface $=\sqrt{h^2+r^2}=\sqrt{12^2+5^2}=\sqrt{144+25}=\sqrt{169}=13$ cm.

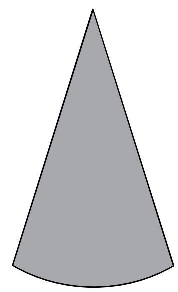
The area of all the faces $=10\times 10+\frac{1}{2}(4\times 10)\times 13=100+260=360$ sq. cm.

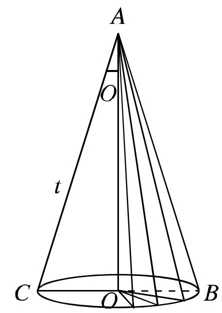
And its volume $=\frac{1}{3}\times(10\times10)\times12=10\times10\times4=400$ cubic centimres. Therefore, the area of all surfaces of the pyramid is 360 sq. cm. and the volume is 400 cubic centimetres.

Activities:
1. Draw a regular and an irregular (a) prisms (b) pyramids.
2. If possible, fins the total surface area and the volume of the solids drawn by you.

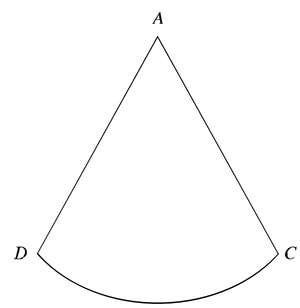
4. Right circular cone

The solid formed by a complete revolution of a right-angled triangle about one of its sides adjacent to the right angle as axis is called a right circular cone. In the figure, the right circular cone ABC is formed by revolving the right-angled triangle OAC about OA. In this case, if θ is the vertical angle of the triangle then it is called the semi-vertical angle of the cone.





If the circular cone has height OA (=h), radius of the base OC (=r) and slant height AC (=l), then



(a) Area of the curved surface

$=\frac{1}{2}\times$ circumference of the base $\times$ slant height .

$=\frac{1}{2}\times 2\pi r\times l=\pi rl$ square units

(b) Area of the whole surface

=area of curved surface + area of base

$=\pi rl+\pi r^2=\pi r(l+r)$ square units.

(c) Volume $=\frac{1}{2}\times$ area of base $\times$ height $=\frac{1}{3}\times\pi r^2 h$ cubic units.

[You will learn the method of deduction of this formula in higher class]

Example 4. If a right circular cylinder has a height of 12 cm. and a base of diameter 10 cm., then find its slant height, the area of the curved surface and the whole surface and its volume .

Solution: Radius of the base $r=\frac{10}{2}$ cm. = 5 cm.

Therefore, slant height $=\sqrt{h^2+r^2}$ cm. $=\sqrt{12^2+5^2}$ = 13 cm.

Area of the curved surface = $\pi rl=\pi\times5\times13$

= 204.2035 square centimetres.

Area of the whole surface = $\pi r(l + r) = \pi \times 5(13+5)$

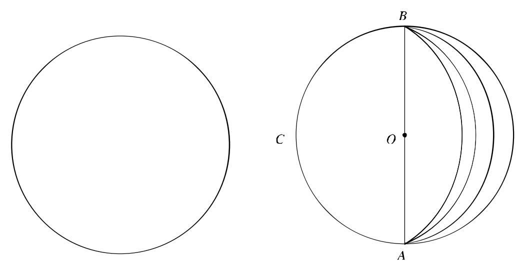
$= 282.7433$ square centimetres.

Volume = $\frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 5^2 \times 12 = 314.1593$ cubic centimetres.

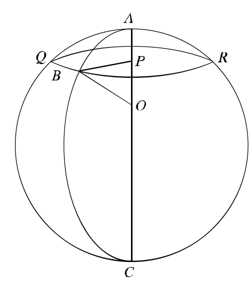
Activities: Collect a conical cap used in birthday or any ceremonial parties and find the volume and the area of its curved surface.

5. Sphere

The solid formed by a complete revolution of a semi-circle about its diameter as axis is called a sphere. The centre of the semi-circle is the centre of the sphere.



The surface formed by the revolution of the semi-circle about its diameter is the surface of the sphere. The straight line drawn from the centre to the surface of the sphere is the radius of the sphere.



The centre of the sphere CQAR is the point O, radius OA = OB = OC and a plane perpendicular to OA and passing through a point at a distance h from the centre cuts the sphere and form the circle QBR. The centre of this circle is P and radius PB. Then PB and OP are perpendicular to each other.

$\therefore \quad OB^2 = BP^2 + OP^2$

i.e. $PB^2 = OB^2 - OP^2 = r^2 - h^2$

If the radius of the sphere is r then

(a) Area of the surface of the sphere = $4\pi r^2$ square units.

(b) Volume = $\frac{4}{3}\pi r^3$ cubic units

(c) Radius of the circle formed by the section of a plane at a distance h from the centre = $\sqrt{r^2 - h^2}$ units.

Activities: Find the radius of a toy ball or a football. Hence find its volume.

Eample 5: An iron sphere of diameter 4 cm. is flattened into a circular iron sheet of thickness $\frac{2}{3}$ cm. What is the radius of the sheet?

Solution: Radius of the iron sphere = $\frac{4}{2}$ cm. = 2 cm. $\therefore$ its volume = $\frac{4}{3}\pi.2^3 = \frac{32}{3}\pi$ cubic centimetres.

Let the radius of the sheet = r cm. The thickness of the sheet $=\frac{2}{3}$ cm.

$\therefore$ Volume of the sheet = $\pi r^2 \times \frac{2}{3}$ cubic centimetres $=\frac{2}{3}\pi r^2$ cubic centimetres.

By the given condition, $\frac{2}{3}\pi r^2 = \frac{32}{3}\pi$ or, $r^2 = 16$ or, $r = 4$

$\therefore$ Radius of the sheet = 4 cm.

Example 6: A right circular cone, a semi-sphere and a cylinder of equal heights stand on equal bases. Show that their volumes are in the ratio 1: 2: 3.

Solution: Let the common height and the radius of the equal bases be h and r units respectively. Since the height of a semi-sphere is equal to its radius.

$\therefore$ Volume of the cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3$ cubic units.

Volume of the semi-sphere = $\frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$ cubic units.

Volume of the cylinder = $\pi r^2 h = \pi r^3$

$\therefore$ Required ratio $=\frac{1}{3}\pi r^3 : \frac{2}{3}\pi r^3 : \pi r^3 = \frac{1}{3} : \frac{2}{3} : 1 = 1 : 2 : 3$

Eaxample 7: The length, breadth and height of a rectangular block of iron are respectively 10, 8 and $5\frac{1}{2}$ cm. How many spherical shots of radius $\frac{1}{2}$ cm. can be made by melting the block?

Solution: Volume of the iron block = $10 \times 8 \times 5\frac{1}{2}$ cubic cm. = 440 cubic cm.

Let the required number of shots = n

$\therefore$ Volume of n shots = $n \times \frac{4}{3}\pi\left(\frac{1}{2}\right)^3 = \frac{n\pi}{6}$ cubic cm.

By the condition of the question, $\frac{n\pi}{6} = 440$ $\therefore n = \frac{440 \times 6}{\pi} = 840.34$

$\therefore$ Required number of shots is 840

Example 8: If the volume of a right circular cone is V, the area of its curved surface is S, radius of the base is r, height is h and semi-vertical angle is α. Then show that (i) $S = \frac{\pi h^2 \tan\alpha}{\cos\alpha} = \frac{\pi r^2}{\sin\alpha}$ square units.

$(ii)\ V = \frac{1}{3}\pi h^3 \tan^2 \alpha = \frac{\pi r^3}{3\tan\alpha}$ cubic units.

Solution: In the adjacent diagram, height of the cone $OA = h$, slant height $AC = l$, radius of the base $OC = r$, the semi-vertical angle $\angle OAC = \alpha$.

Here slant height $l = \sqrt{h^2 + r^2}$.

From the diagram, it is seen that $\tan\alpha = \frac{r}{h}$

$\therefore\ r = h\tan\alpha$ or, $h = \frac{r}{\tan\alpha} = r\cot\alpha$

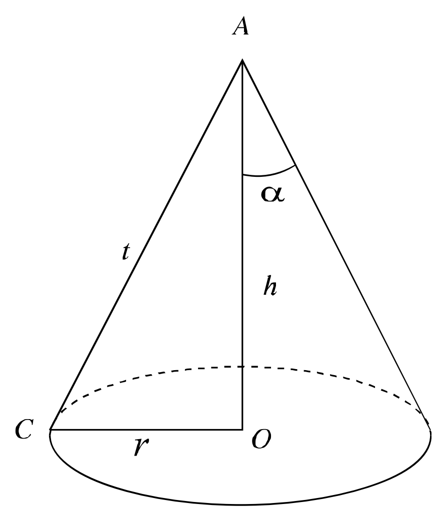
Now $(i)\ S = \pi r l = \pi r\sqrt{h^2 + h^2\tan^2\alpha} = \pi r h\sqrt{1+\tan^2\alpha} = \pi r h\sqrt{\sec^2\alpha}$

$$= \pi r h \sec\alpha = \frac{\pi r}{\cos\alpha}.r\cot\alpha = \frac{\pi r^2}{\cos\alpha}.\frac{\cos\alpha}{\sin\alpha} = \frac{\pi r^2}{\sin\alpha} \text{ square units.}$$

$(ii)\ V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(h\tan\alpha)^2 h = \frac{1}{3}\pi h^3\tan^2\alpha$

$$= \frac{1}{3}\pi\left(\frac{r}{\tan\alpha}\right)^3 \tan^2\alpha = \frac{\pi r^3}{3\tan\alpha} \text{ cubic units.}$$

6. Compound solid



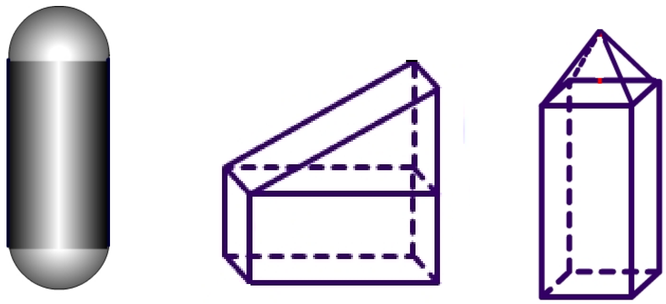
A solid body consisting of two solids is a compound solid. Exaples of a few compound solids are:

(1) A prism placed on a rectangular solid is a compound solid when both of them have identical surface.

(2) If the base of a triangular prism and that of a regular tetrahedron are identical, they can be used to make a compound solid.

(3) If the radius of a sphere and that of the base of a cone are equal, they can form together a compound solid.

(4) A capsule is a compound solid having two identical hemispheres at two ends of a circular cylinder with base radius equal to the radius of each hemisphere.



Compound solids of different shapes

In this way, two or more solids can be combined together to make a compound solid. Many beautiful architectural constructions are compound solids. The items you use for exercises are made of such compound solids.

Activities: Draw and describe a compound solid on your own. Can you write down formula for its surface area and volume?

Example 9. The length and radius of a capsule is 15 cm. and 3 cm. respectively. Find the volume and total area of surfaces of the capsule.

Solution: The length of the capsule is 15 cm. Since the two ends of the capsule are hemispherical, the length of its cylindrical part $l = 15 - (3+3) = 9$ cm.

Therefore, the total area of surfaces of the capsule = area of the surfaces of two hemispheres + area of the surface of the cylinder cm.

$= 2\times\frac{1}{2}\times 4\pi r^2 + 2\pi rl = 4\pi(3)^2 + 2\pi\times 3\times 9$ [$\because$ $r = 3$ cm.]

$= 90\pi = 282.74$ sq. cm.

Volume of the capsule

$= 2\times\frac{1}{2}\times\frac{4}{3}\pi r^3 + \pi r^2 l = \frac{4}{3}\pi(3)^3 + \pi(3)^2\times 9 = 117\pi = 367.57$ cubic cm.

Exercise 13

1. What is the length of a diagonal of a rectangular parallelopiped whose length, breadth and height are respectively 8 cm., 4 cm. and 3 cm.?

   a. $5\sqrt{2}$ cm. b. 25 cm.

   c. $25\sqrt{2}$ cm. d. 50 cm.

2. Lengths of other two sides except the hypotenuse of a right-angled triangle are 4 cm. and 3 cm. If the triangle is revolved about the larger side, the evolved solid will be a

   (i) right circular cone

   (ii) right circular cylinder

   (iii) the area of the base of the evolved solid is 9π square centimetres.

Which one of the above sentences is correct?

a. i b. ii

c. i & iii   c. ii & iii

Answer to the questions 3 and 4 according to the informations given below.

A spherical ball of diameter 2 cm. exactly fits in a cylindrical box

3. What is the volume of the cylinder?

a. 2π cc.   b. 4π cc.

c. 6π cc.   d. 8π cc.

4. What is the volume of the unoccupied portion of the cylinder?

a. $\frac{\pi}{3}$ cc.   b. $\frac{2\pi}{3}$ cc.

c. $\frac{4\pi}{3}$ cc.   d. $\frac{3\pi}{3}$ cc.

Answer to the questions 5 and 6 according to the following informations:

A metallic solid sphere of diameter 6 cm. is melted and a right circular cylinder the radius of whose base is 3 cm. is made.

5. What is the height of the cylinder made?

a. 4 cm.   b. 6 cm.

c. 8 cm.   d. 12 cm.

6. What is the area of the curved surface of the cylinder in square centimetres?

a. 24π b. $42\ \pi$

b. 72π d. 46π

7. The length, breadth and height of a rectangular parallelolepiped are respectively 16 metres 12 metres and 4.5 metres. Find the area of its surfaces, volume and length of a diagonal.

8. A rectangular tank of length 2.5 metres, breadth 1.0 metre stands on the ground. If its height is 0.4 metre, find the volume and its area of the interior surface.

9. Find the area of the whole surface of the cube whose edge is equal to the diagonal of the rectangular solid whose dimension are 5 cm., 4 cm. and 3 cm.

10. A hostel building is to be constructed for 70 students such that each student requires 4.25 square metres of floor and 13.6 cubic metres of space. If the hostel room is 34 metres long, what will be its breadth and height?

11. If the height of a right circular cone is 8 cm and the radius of its base is 6 cm, find the area of the whole surface and the volume.

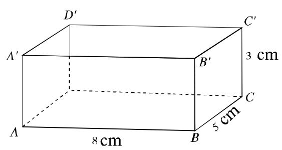
12. The height of a right circular cone is 24 cm. and its volume is 1232 cubic cm. What is its slant height?

13. The length of two sides at right angle of a right-angled triangle is 5 cm. and 3.5 cm. Find the volume of the solid formed by revolving it about its greater side.

14. Find the surface and volume of a sphere of radius 6 cm.

15. Three spherical balls of glass of radii 6, 8 and r cm. are melted and formed into a single solid sphere. Find the value of r.
16. The outer diameter of a hollow sphere is 13 cm. and the thickness of the iron is 2 cm. A solid sphere is formed with the iron used in the hollow sphere. What will be its diameter?
17. A solid sphere of radius 4 cm. is melted and formed into a uniform hollow sphere of outer radius 5 cm. Find the thickness of the second sphere.
18. The radius of a solid sphere of iron is 6 cm. With the iron contained in it, how many solid cylinders of length 8 cm. and diameter of the base 6 cm. can be formed?
19. A spherical ball of circumference cm. exactly fits into a cubical box. Find the volume of the unoccupied portion of the box.
20. A sphere of radius 13 cm. is cut off by a plane perpendicular to its diameter through a point at a distance 12 cm. from the centre. Find the area of the plane formed.
21. The outer length, breadth and height of a wooden box with top are respectively 1.6, 2 and 8 metres and its wood is 3 cm. thick. What is the area of the inner surface of the box? What will be the cost of painting the inner surface of the box at the rate of Tk. 14.44 per square metre?
22. How many bricks each of length 25 cm., breadth 12.5 cm. and 8 cm. will be required to construct a wall of height 2 metres and thickness 25 cm. around a rectangular garden of length 120 metres and breadth 90 metres?
23. The length and breadth of a rectangular solid are in the ratio 4 : 3 and its volume is 2304 cubic centimetres. The total cost of making a lead coating at the bottorn of the solid at Tk. 0.10 per square centimetre. is Tk. 1920. Find the dimensions of the solid.
24. A conical tent has a height of 7.50 metres. How much canvas will be required if it is desired to enclose a land of 2000 square metres?
25. Lengths of two sides of a prism with a pentagonal base are 6 cm. and 8 cm. and the length of each of the other three sides is 12.5 cm. Find the total area of the surfaces and the volume of the prism.
26. The height of a regular hexagonal prism having side of 4 cm. is 5 cm. Find the total area of surfaces and the volume.
27. A pyramid is situated on a regular hexagon of side 6 cm. and its height is 10 cm. Determine the total area of surfaces and the volume.
28. If the length of an edge of a regular tetrahedron is 8 cm., find the total area of surfaces and the volume.

29. The lower part of a construction is a parallelepiped of length 3 metres and the upper part is a regular pyramid. If the base of the pyramid is of side 2 metres and the height is 3 metres, find the total area of surfaces and the volume of the construction.
30. A godown of two-part roof is constructed on a land of length 25 metres and breadth 18 metres. Its height is 5 metres. Each part of the roof is of 14 metres wide. Find the volume of the godown.
31.



a. Find the total area of surfaces of the solid mentioned in the diagram.
b. Find the approximate number of solid spheres which can be made after melting a cube whose side is equal to the diagonal of the solid.
c. A rectangle of size of ABCD surface of the solid is revolved about the larger side . Find the volume and the total surface area of the solid formed.

32. The diameter of the base of a tent like a right circular cone is 50 metres and its height is 8 metres.
a. Find the slant height of the tent.
b. How much land in square metres will be required to construct the tent? Find the volume of the vacuum space inside the tent.
c. What will be the cost of the canvas of the tent if its price is Tk. 125 per square metre?

Chapter Fourteen
Probability

In our everyday life we frequently use the word 'probability.' For example passing rate of SSC examination in this year is poor. Tomorrow the temperature will probably rise by 2 degrees celcius ; The probability of raining tomorrow is slight ; The probability of Bangladesh winning the Asia cup is high.

In such statements we indicate the likelihood of some event without assigning a numerical measure, or value, to the likelihood, that is the probability of the event. The purpose of this chapter is to explain how we can assign a numerical value to the probability of the occurrence or non-occurrence of an event.

After completing the chapter, students will be able to –

- Explain the concept of probability
- Give examples of certain events, impossible event and possible event
- Describe the consequences of repetition of an event
- Find the probability of an event happening repeatedly
- Solve simple real-life problems involving probability

14.1 Basic Concepts Related to Probability

Random Experiment

The word 'experiment' is used here in a very wide sense. Tossing a coin is an experiment, which has two mutually exclusive outcomes : Head (H) or tail (T). Throwing of a dice is an experiment ; it has six possible outcomes ; the face up may have any of the six umber 1, 2, 3, 4, 5, 6 on it. These are examples of random experiments.

Event : The outcome of a random experiment is called an event. For example getting 3 on the throw of a dice is an event. Getting head on the tossing of a coin is an event.

Equally Likely Events :

If the outcomes of a random experiment are such no outcome is more or less likely to happen than any other outcome, than the possible outcomes are called equally likely events. For example, in the tossing of a coin the occurrence of head or tail are equally likely events, unless the coin is defective in some way. We then may that the coin is 'unbiased.'

Mutually Exclusive Events :

Two or more possible outcomes of a random experiment are called mutually exclusive event if the occurrence of one of those events, precludes the possibility of the other events. In the tossing of a coin, the occurrence of head and tail are mutually exclusive events ; for if head occurs then tail cannot occur and vice versa.

Sample Space and Sample Point :

The set of all possible outcomes of a random experiment is called the sample space. The tossing of a coin has two possible outcomes ; H (head) and T (tail). So in this case the sample space is S = {H, T}. Suppose two coins are tossed simultaneously (by two persons). The possible outcomes are : HH (head on both coins) and TH (tail on the first coin and head on the second coin). TT (tail on both coins), HT (head of the first coin, tail on the second coin) and TH (tail on the first coin and head on the second coin). So in this case the sample space is S = {HH, HT, TH, TT}.

Every element of a sample space is called a sample point. In the preceding example, the sample consists of four sample points.

14.2 Determination of logic-based probabilities

Example 1. Suppose an unbiased dice is thrown. What is the probability of getting 5?

Solution : The possible of the throwing of a dice are : 1, 2, 3, 4, 5, 6. The dice is unbiased means, these outcomes are all equally likely. So the possibility of occurring of a particular is one-sixth. We write

$$P(5)=\frac{1}{6}$$

And say that the probability of getting 5 (or any other number) on the throw of a dice is $\frac{1}{6}$.

Example 2. What is the probability of getting an even number on the throw of an unbiased dice ?

Solution : The possible outcomes of a throw of a dice are 1, 2, 3, 4, 5, 6. The even number among these numbers are 2, 4, 6 ; so there are three outcomes the occurrence of any of which will give us an even number on the dice. We say there are 3 favourable outcomes, of all which are equally likely. Since there 6 possible, equally likely outcomes, the probability of getting an even number on the dice is $\frac{3}{6}$.

$$\therefore \text{ P(even number)} = \frac{3}{6}=\frac{1}{2}.$$

So we can formularize the following definition of the probability of an event E :

$$P(E)=\frac{\text{number of cases favourable to the event E}}{\text{total number of possible outcomes}}$$

If there no case favourable to an event E, then ($P(E)$ = 0 ; if all possible outcomes are favourable to the event E, then P(E) = 1. For example, the probability of getting a head or tail in the tossing of a coin is 1. The probability of getting an even or an odd number in the throw of a dice is 1, because every number is either even or odd.

14.3 Two special event

Certain Events :

An event which is sure to occur is called a certain event. The probability of any certain event is 1. For instance, the probability that the sun will rise tomorrow from the east is 1. The probability that the sun will set this evening in the west is likewise 1. The probability of not seeing sun in the night is also 1. The probability of seeing in the night is also 0.

Impossible Events :

An event which is sure not to occur is called an impossible event. The probability of an impossible event is 0.

For instance, the probability that tomorrow the sun will rise in the west or set in the east, is 0. Similarly, the probability of getting 7 in the throw of a dice is 0. Each of these events is an impossible event.

Example 3. In a bag there are 4 red, 5 white and 6 black balls. A ball is chosen at random. What is the probability that the ball will be

(i) red (ii) white (iii) black

Solution : There are in all 4 + 5 + 6 = 15 balls in the bag. So the total number of favourable cases is 15.

The event of drawing a red, white or black ball be denoted by R.W.B respectively.

(i) There are 4 red balls in the bag ; so there are 4 favourable cases for the event

$$\therefore\ P(R) = \frac{\text{number of events favourable to R}}{\text{total number of outcomes}} = \frac{4}{15}$$

(ii) There are 5 white balls in the bag ; so there are 3 favourable cases for the event W.

$$\therefore\ P(W) = \frac{5}{15} = \frac{1}{3}$$

(iii) There are 6 black balls in the bag ; so there are 6 favourable case for the event B.

$$\therefore\ P(B) = \frac{6}{15} = \frac{2}{5}$$

Activity :

1. An unbiased dice is thrown. Find the probability of getting
   (i) 4 (ii) 4 or a number greater than 4 (iii) a number less than 5.
2. A bag contains 6 black, 5 red and 8 white marbles. A marble is drawn at random from the bag. Find the probability that the marble is
   (i) red (ii) black (iii) yellow (iv) not black.

11.4 Data Based Probability

In the logical approach to probability the outcomes are assumed or required to be equally likely. Moreover, in many situations, there is no thing concrete to which the logical approach may be applied. For Example, today's says that the probability of raining today is 30% that is $\frac{30}{100}=\frac{3}{10}$: or that Brazil winning the world football cup is 40% = $\frac{40}{100}=\frac{2}{5}$: or that the probability of Bangladesh winning in Asia cup is 60% = $\frac{60}{100}=\frac{3}{5}$. Then statement are based on statistics or data of past games or rainfall records ; probability derived from past experience or data is called data-based or a posteriori probability. In contrast, probability based on logic is called a priori probability.

Suppose a coin is tossed 1000 times which resulted in head in 523 tossings. So the relative frequency of getting head is $=\frac{523}{1000}=0\cdot 523$.

The more times this experiment (tossing of a coin), the more accurate will be the relative frequency to a number that will indicate the probability of getting head in a single tossing of the coin. This what is called databased probability.

Example 4. According to metrological records, in the month of July last year it rained on 21 days. What is the probability that it will rain on fourth of July this year?

Solution : Data revels that out of the 31 days in the month of July, it rained on 21 days last year. So the probability of raining on a particular day of July this year, is $\frac{21}{31}$.

Example 5. In a servey among readers of newspaper it was found that 65 persons read Prothom Alo, 40 persons read Bhorer Kagaj, 45 read Janakantho, 52 read Jogantor. If one person is chosen at random from amongst there reader, what is the probability that the person reads Jogantor? What is the probability that the person does not read Prothom Alo?

Solution : The total number of readers is 65 + 40 + 45 +52 = 202, of these 202 persons, 52 read Jogantor.

So, the probability that the chosen person reads Jogantor is $\frac{52}{202}=\frac{26}{101}$. 65 persons read Prothom Alo. So (202 – 65) = 137 persons do not read Prothom Alo. So that the chosen person does not read Prothom Alo is $\frac{137}{202}$.

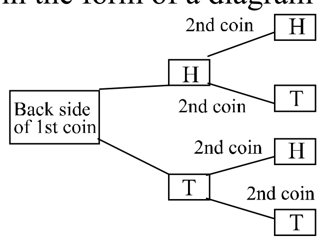
Activity : In a survey among the newly admitted students, it is found that 284 students have taken Economics, 106 have taken History, 253 have taken Sociology, 169 have taken English. A newly admitted student is chosen at random. What is the probability that the student has not taken Sociology ?

14.5 Determination of Probability by Sample and Probability Tree

We have already mentioned that the set of all possible outcomes of an experiment is called the sample space. Often the sample is quite large. In such cases, the counting of all sample points and the formation of the sample space may be cumbersome and may lead to mistakes. In such cases we may build the sample space in the shape of a tree, called the probability tree, and use it to find the probability of various events.

Example 6. Suppose two unbiased coins are tossed together. Form the sample space. Find the probability of getting H on the first coin and 7 on the second coin.

Solution : The tossing of two coins may be treated as a two step process. In the first step a coin is tossed which may in result either H or T. In the second the other coin is tossed which may, likewise, result in either H or T. All possible outcomes this experiment may be exhibited in the form of a diagram as follows :



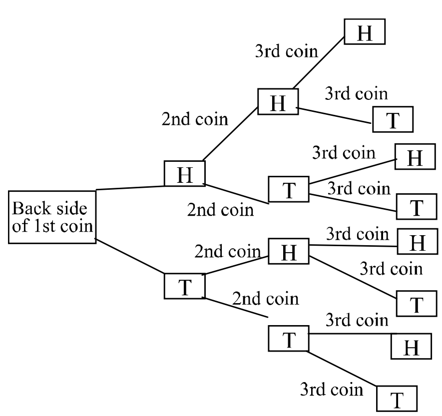
So the sample points are : HH, HT, TH, TT ;
and the sample space is {RH, HT, TH, TT}.

There are 4 sample points and each point (outcome of the experiment) is equally likely. So the probability of getting H on the first coin and T on the second coin is $P(HT) = \frac{1}{4}$.

Example 7. An unbiased coin is tossed thrice, Form the sample space as a probability tree and exhibit it as a set. Hence find the probability of each of the following events.

(i) getting just one tail
(ii) getting head in all three tossings
(iii) getting at least one tail.

Solution : In each throw the possible outcomes are H, T. As there are three throw the probability tree here will be on extension of that in the preceding example as shown below.



The sample space is S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} is set of a sample points. The probability of any one of these outcomes is $\frac{1}{8}$.

(i) The sample points (outcomes) containing just one tail are :

THH, RHT, RTH. These are three outcomes out of 8 possible outcomes.

$\therefore$ P (just one tail) = $\frac{3}{8}$

(ii) Getting head in all three tossings is the outcome HHH only.

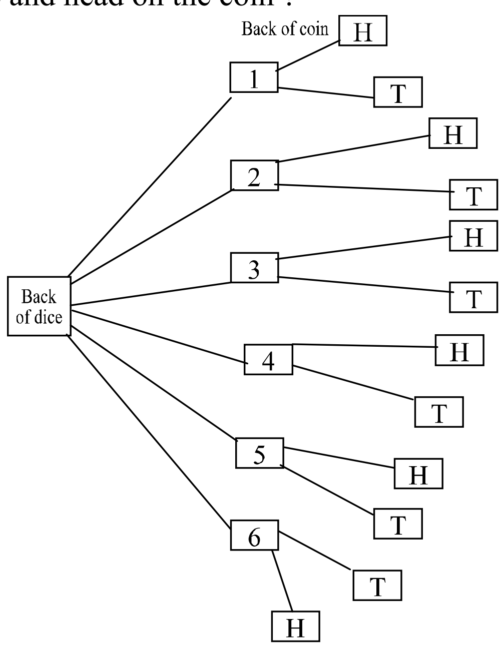
$\therefore$ P(HHH) = $\frac{1}{8}$.

(iii) The sample points containing at least one tail are all except HHH.

$\therefore$ P (at least tail) = $\frac{7}{8}$.

Note carefully the distinct between 'just one tail' and 'at least one tail'. If the question had asked the probability of one tail, then there would a chance of misunderstanding, though logically one tail means 'at least one tail'.

Example 8. An unbiased dice and an unbiased coin are thrown and tossed once. Form the probability tree and write down the sample space. What is the probability of getting 5 on the dice and head on the coin ?

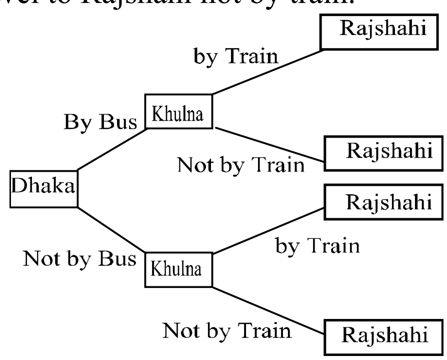


Solution : Consider that the dice is thrown first, This will result in one of the six possible outcomes : 1, 2, 3, 4, 5, 6. In the second, the tossing of the coin will result in two possible outcomes : H, T. The corresponding probability tree is shown below :

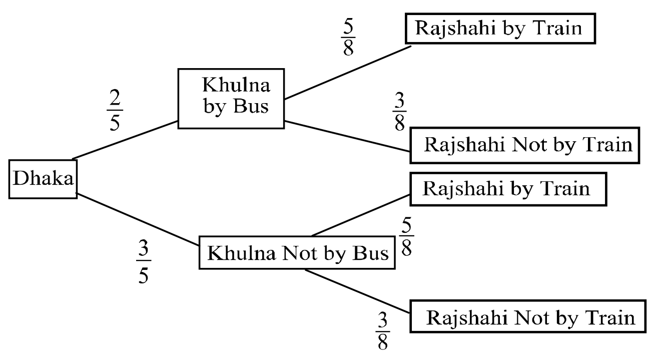
The sample space is S = {1H, IT, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T} is a set of 12 sample points.

$\therefore$ the probability of getting 5 on the dice and Head on the coin is $P(5H) = \frac{1}{12}$.

Example 9. The probability that a person will travel from Dhaka to Khulna by bus is $\frac{2}{5}$ and that he will travel from Khulna to Rajshahi by train is $\frac{5}{8}$. Use a probability tree to determine the probability that the person will from Dhaka – Khulna by bus and will subsequently travel to Rajshahi not by train.



First we list the possible choice of the mode of transport from Dhaka to Khulna and subsequently from Khulna to Rajshahi.



Next we list probability of each choosing mode of transportation and thus construct the probability tree.

Therefore, the probability that the person will travel from Dhaka to Khulna by bus and then from Khulna to Rajshahi not by train is P(by bus to Khulna and by train to Rajshahi) = $\frac{2}{5} \times \frac{3}{8} = \frac{6}{40} = \frac{3}{20}$.

Activity :

1. Form a probability three showing all possible outcomes of three tossings of a coin and write down the sample space. Hence find the probability of getting

(i) the same outcome in all three tossings (ii) at least two tails (iii) at most two tails. 2. Form a probability tree for throwing of one dice and two coins.

Exercise 14

1. Which is the probability of getting 3 in the throw of a dice ?

   (a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{2}$

Answer questions 2 and 3 based on the given information.

A ball is drawn at random from a bag containing 12 blue, 16 white and 20 black balls.

2. What is the probability that the ball is blue ?

   (a) $\frac{1}{16}$ (b) $\frac{1}{12}$ (c) $\frac{1}{8}$ (d) $\frac{1}{4}$

3. What is the probability that the ball is not white ?

   (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{16}$ (d) $\frac{1}{48}$

A coin is tossed thrice. Answer Questions 4 and 5.

4. What is the probability of getting head most of times ?

   (a) 1 time (b) 2 times (c) 3 times (d) 4 times

5. What is the probability of getting T the least number of times ?

   (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

6. According to report of Chittagong meteorological office, in the first week of July 2012 it rained on 5 days. What is the probability of not raining on 8^{th} July ?

   (a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{5}{2}$ (d) 1

7. Thirty tickets are numbered serially form 1 to 30. The tickets are mixed thoroughly and one ticket is drawn at random. Find the probability that drawn is (i) an even number (ii) divisible by 4 (iii) less than 8 (iv) greater than 22.
8. In a certain lottery 570 tickets have been sold. Rahim has bought 15 tickets. The tickets are missed together thoroughly and one ticket is drawn at random for the first price. What the probability of Rahim getting the first prize.
9. What is the probability of getting an even number or a number divisible by 3 in a single throw of a dice ?

10. According to report of a rural health centre, in a certain period, 155 babies were born whose weight less than the normal weight, 386 babies were born with normal weight and 98 babies were born with more that normal weight. One baby is selected at random from amongst these babies ; what the probability that the baby was born with more than normal weight ?

11. Out of 2000 newly licensed drivers
A survey was carried about violation of traffic rules by. The result in shown in the table below :

Number of Violation of Traffic Rules	**Number of Drivers**
0 1 4 5 or more	1910 46 18 12 9 5

Out of these 2000 drivers one driver is chosen at random. What is the probability that the driver has violated traffic rules
(i) once
(ii) more than four times ?

12. The employers of a certain factory are classified into four categories and the number of employees of each category are given in the table below :

Classification Number of	**Employees**
Managerial	157
Inspection	52
Production	1473
Office work	215

One employee is chosen at random ; what is the probability that the person is in managerial position ? What the probability that the person is engaged in managerial or production work ? What is the probability that the person is not engaged in production ?

13. Form a probability tree for the successive tossing throw of a coin and of a dice.
14. Fill up the following table with the help of probability tree.

Tossing of the coin	**All possible outcomesProbability**
One tossing of the coin Two tossings of the coin Three tossings of the coin	P(T) = P(1H) = P(HT) = P(HHH) = P(2H) =

15. The probability that a certain person will travel from Dhaka to Rajshahi by train is $\frac{5}{9}$ and that subsequently the person will travel to Khulna by bus is $\frac{2}{7}$. Use a probability tree to find the probability that the person will travel to Rajshahi not by train and then travel to Khulna by train. Find also the probability that the person will travel to Rajshahi by train and then travel to Khulna not by bus.
16. The probability that a certain person will travel from Dhaka to Rajshahi by train is $\frac{2}{9}$, the probability that the person take a flight is $\frac{1}{9}$. The probability that subsequently the person will travel to Khulna by bus is $\frac{2}{5}$ and the probability that the person will travel by train is $\frac{3}{7}$. Use a probability tree to find the probability that he will travel by train to Rajshahi and then by bus to Khulna.

17. A two-taka coin is tossed four times. Denote its side with flower by L and the side child by C.
    a. If the coin is tossed twice rather than four times, what is the probability of getting a L and that of not getting a C ?
    b. Draw the probability tree and write down the sample space.
    c. Show that in n times tossing of the coin the sample space will consist of 2^n points.

Answer

Exercise 1.1

1 – 4 Do yourself :

5. (*a*) $A = \{3, 5, 7, 9, 11, 13, 15, 15, 17, 19\}$

$B = \{3, 5, 7, 9, 11, 13, 17, 19\}$

(*b*) $C = \{3, 5, 7, 9, 11, 13, 17, 19\}$

$D = \{3, 5, 7, 9, 11, 13, 15, 17, 19\}$

6. (*a*) 8 (*b*) 56 and 24

7. (*a*) 12 (*b*) 21, 21

8. 7, 4 9. 0, 3 18. 2 persons 19. (a) 4, (b) 6 (c) 28

20. (*a*) 4 (*b*) 6 (*c*) 28 21. (*a*) 4 (*b*) 16, 7 (*c*) 9

23. (*a*) $A \cap B = \{x : 2 < x < 3, \quad x \in R\}$

(*b*) $A \cap B = \{x : 1 \leq x \leq 3, \quad x \in R\}$

24. (*a*) $A' \cap B = \{x : 4 < x < 6\}$ (*b*) $A \cap B' = \{x : 1 < x < 3\}$

(*c*) $A' \cap B' = \{x : x \leq 4 \text{ or } x \geq 6\}$ 26. (*i*) 10% (*ii*) 50%

Exercise 1.2

7. (*a*) Dom S = {1, 2, 3, 4}, Range S = {5, 10, 20}.

S^{-1} = {(5, 1), (10, 2), (15, 3), (20, 4)}

(b) S and S^{-1} are functions

(c) one-one function.

b. (*a*) Dom S = {–3, –2, –1, 0, 1, 2, 3}, Range S = {–1, 0, 3, 8}

S^{-1} = {(8, –3), (3, –2), (0, –1), (–1, 0), (3, 2), 8, 3)},

(b) S is function S^{-1} is not function as (0, 1) and (1, –1), $\leftarrow$ S^{-1}

(c) one-one function

c. (*a*) Dom S = $\left\{\frac{1}{2}, 1, \frac{5}{2}\right\}$, Range S = {–2, –1, 0, 2}.

S^{-1} = $\left\{\left(0, \frac{1}{2}\right), (1,1), (-1,1), \left(2, \frac{5}{2}\right), \left(-2, \frac{5}{2}\right)\right\}$

(b) S is not functions (1, 1) and (1, –1), $\leftarrow$ S^{-1} function

(c) one-one function

d. (*a*) Dom S = {-3, 1, 0, 3}, Range S = {–3, –1, 0, 3}

S^{-1} = S

(b) S, S^{-1} functions

(c) is not one-one function

e. (*a*) Dom S = {2}, Range S = {1, 2, 3}

S^{-1} = {(1, 2), (2, 2), (3, 2)}

(b) S is not function

(c) is not one-one function

8. (a) Dom F = R, one-one

(b) Dom F = R, is not one-one

(c) Dom F = {x ← R : x ≥ 1}, one-one

[as $\sqrt{x-1}$ means nonnegative square root.]

(d) Dom F = R / {2}, one-one

(e) Dom F = R, is not one-one

(f) f = R, one-one

9. (a) 0, 2, 3 (b) [a] (c) 26 (d) 1, + y^2

10. (a) 3, 1, 0, 1, 3 (b) ± 4 (c) 0 (d) y

11. (a) Dom F = R (b) Range R = R (c) $F^{-1} = R \to R$ (d) $F^{-1}(x) = \sqrt{x}$

19, (a) P = {(2, 2), (–1, 3), (1, 5), (1, 7), (–2, 5)

(a) S = {(–2, 4), (–1, 1), (0, 0), (1, 1), (2, 4)}

(b) S and S^{-1} are functions

(c) one-one function

Dom S = {–2, –1, 0, 1, 2}, Range S = {0, 1, 4}

20. (a) $F(x+1) = 2x+1, F\left(\frac{1}{2}\right) = 0$

(b) one-one function.

21. (a) Set of Positive real number R_+ Dom F = R_+

(b) Image are different for different values of the domain of the function F(x) is not function.

(c) Range F = R_+, x = 10.

Exercise 2

6. (a) $Q(x) = x^{n-1} + ax^{n-2} + a^2x^{n-3} + a^3x^{n-4} + ... + a^{n-1}$

(b) $Q(x) = x^{n-1} - ax^{n-2} + a^2x^{n-3} - a^3x^{n-4} + ... + (-1)^{n-1}a^{n-1}$

7. $Q(x) = x^{n-1} - ax^{n-2} + a^2x^{n-3} - a^3x^{n-4} + ... + (-1)^{n-1}a^{n-1}$

9. (*i*) $(x+1)^2(x+2)(x+3)$

(*ii*) $(2a-1)(a+1)(a+2)(2a+1)$

(*iii*) $(x+1)(x^2+x+1)$

(*iv*) $(x+y+z)(xy+yz+zx)$

(*v*) $-(x-y)(y-z)(z-x)$

(*vi*) $-(a-b)(b-c)(c-a)(a+b)(b+c)(c+a)$

12. (*a*) 1 (*b*) $\dfrac{x}{(x-a)(x-b)(x-c)}$ (*c*) 0 (*d*) $\dfrac{1}{x-1}$

13. (*a*) $\dfrac{2}{x}+\dfrac{3}{x+2}$ (*b*) $\dfrac{6}{x-4}-\dfrac{5}{x-3}$ (c) $\dfrac{1}{x}-\dfrac{2}{x-2}+\dfrac{2}{x+3}$

(*d*) $x+\dfrac{1}{x-1}+\dfrac{2}{x+3}$ (*e*) $\dfrac{3x+1}{x^2+4}-\dfrac{2}{x+1}$ (*f*)

$\dfrac{4}{10(2x+1)}-\dfrac{12}{25(x+3)}-\dfrac{9}{5(x+3)^2}$

Exercise 5.1

1. $3,\ -\dfrac{3}{2}$ 2. $-2-\sqrt{7},\ -2+\sqrt{7}$ 3. $2-\sqrt{3},\ 2+\sqrt{3}$

4. $\dfrac{1}{4}\left(5-\sqrt{33}\right),\ \dfrac{1}{4}\left(5+\sqrt{33}\right)$ 5. $\dfrac{1}{6}\left(-7-\sqrt{37}\right),\ \dfrac{1}{6}\left(-7+\sqrt{37}\right)$

6. $\dfrac{1}{6}\left(9-\sqrt{105}\right),\ \dfrac{1}{6}\left(9+\sqrt{105}\right)$ 7. 4, 4 8. $\dfrac{1}{4}\left(-7-\sqrt{57}\right),\ \dfrac{1}{4}\left(-7+\sqrt{57}\right)$ 9. $\dfrac{1}{3},\ 2$

Exercise 5.2

1. 13 2. $\dfrac{6}{5}$ 3. 9 4. 5 5. 5 6. $\dfrac{5}{2},\dfrac{13}{2},$ 7. 1, 5

8. $2,-\dfrac{9}{2},$ 9. $\dfrac{25}{7},-\dfrac{1}{7}$ 10. $-\dfrac{8}{11},\ -\dfrac{3}{2}$

Exercise 5.3

1. 2 2. $\dfrac{7}{3}$ 3. 6 4. 5 5. 2 6. $\dfrac{5}{2},$ 7. 3 8. 0,

9. 0, 2 10. −1, 0 11. $-\dfrac{1}{21},\ \dfrac{1}{2}$ 12. 2, 3

Exercise 5.4

1. (2, 3), $\left(\dfrac{15}{2},\dfrac{16}{9}\right)$ 2. (3, 4), $\left(-6,\dfrac{5}{8}\right)$ 3. (0, 0), (13, 13), (3, –2), (–2, 3)

4. (0, 0), (5, 5), (2, –1), –1, 2) 5. $\left(\dfrac{1}{5},5\right),\left(\dfrac{4}{5},20\right)$ 6. $\left(3,-\dfrac{5}{3}\right),\left(\dfrac{16}{9},-\dfrac{3}{4}\right)$

7. (1,2), (–1, –2) 8. (7,5), (7,–5) 9. (3, 4), (4, 3), (–3, –4), (–4, –3)

$\left(\sqrt{2},-6\sqrt{2}\right),\left(-\sqrt{2},6\sqrt{2}\right)$

10. (2, 1), 2, –1), (–2, 1), (–2, –1)   11. (1, –2), (2, –1), –1, 2), (–2, 1)

12. (1, 3), (–1, –3), $\left(\frac{13}{\sqrt{21}},\frac{2}{\sqrt{21}}\right),\left(\frac{-13}{\sqrt{21}},\frac{-2}{\sqrt{21}}\right)$

Exercise 5.5

1. 16 metre, 15 metre   2. 13, 9   3. Length 8 metre, breadth 6 metre   4. 19

5. Length 6 metre, breadth 4 metre or Length 16 metre, breadth $1\frac{1}{2}$ metre

6. Length 25 metre, breadth 24 metre   7. Length 8 metre, breadth 6 metre

8. 36   9. $8\sqrt{3}$ metre   10. Length 20 metre, breadth 15 metre

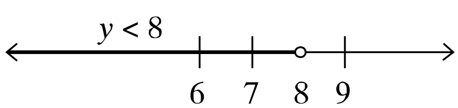
Exercise 5.6

1. (2, 3) 2. (2, 1), $\left(-\frac{1}{2},-\frac{1}{4}\right)$ 3. (4, 0) 4. (1, 2) 5. (3, 3)

6. (2, ± 2), $\left(-2,\pm\frac{1}{2}\right)$ 7. (2, ± 2), $\left(-2,\pm\frac{1}{2}\right)$ 8. (1, 2) $\left(-\frac{1}{3},\frac{2}{3}\right)$

9. (2, ± 2), $\left(-2,\pm\frac{1}{2}\right)$

Exercise 6.1

1.


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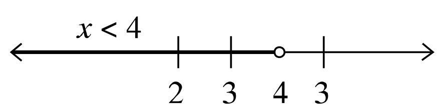

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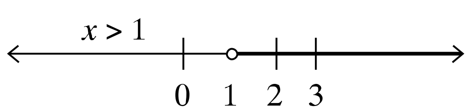

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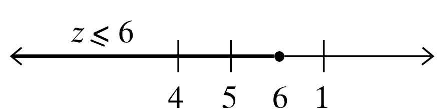

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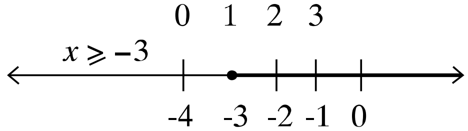

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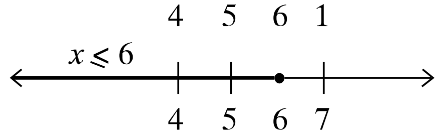
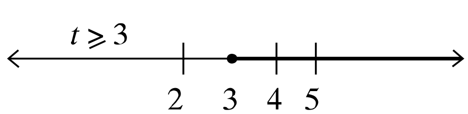
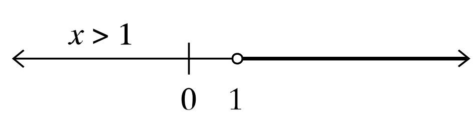


7.


8.


Exercise 6.2

1. $3x+\frac{x+2}{2}<29, 0<x<8$ 2. $4x+x-3\le 40, 0<x<\frac{43}{5}$

3. $0x+20x<500, 0<x<5$ 4. $\frac{x+x+120}{9}\le 100; 0<x\le 390$

5. $5x<40, 5<x<8$ 6. Age of father ≤ 42 years

7. If present age of Geny is x years, $14<x<17$ 8. If time is t second, $t\ge 50$

9. If time of flight is t hour $t\ge 6\frac{1}{4}$ 10. If time of flight is t hour, $t\ge 5\frac{5}{8}$

11. If the number is x, $0<x<5$

Exercise 7

8. (a) $20, 30, 2r$ (b) $5, \frac{15}{2}, \frac{r}{2}$ (c) $\frac{1}{110}, \frac{1}{240}, \frac{1}{r(r+1)}$

(d) $1, 0, 1$ (r is even) and 0 (r is odd)

(e) $\frac{5}{3^9}, \frac{5}{3^{14}}, \frac{5}{3^{r-1}}$ (f) $0, 1, \frac{1-(-1)^{3r}}{2}$

9. (a) $n>10^5$ (b) $n<10^5$ (c) o

11. (a) 2 (b) $\frac{1}{7}$ (c) $\frac{32}{3}$ (d) has no summation (e) $\frac{1}{3}$

12. (a) $\frac{70}{81}(10^n-1)-\frac{7n}{9}$ (b) $\frac{50}{81}(10^n-1)-\frac{5n}{9}$

13. Condition $x<-2$ or $x>0$; summation = $\frac{1}{x}$

14. (a) $\frac{3}{11}$ (b) $\frac{2303}{999}$ (c) $\frac{41}{3330}$ (d) $3\frac{403}{9990}$

Exercise 8.1

1. (a) (*i*) $75^\circ 30'$ (*ii*) $55^\circ 54' 53''$ (*iii*) $33^\circ 22' 11''$
   (b) (*i*) $110^\circ 46' 9\cdot 23''$ (*ii*) $75^\circ 29' 54\cdot 5''$ (*iii*) $55^\circ 54' 53\cdot 35''$

3. 0·2626 m. (approx.)4. 57 km/h (approx.) 5. $\frac{\pi}{5}$ Radian, $\frac{\pi}{2}$ Radian

6. $\frac{2\pi}{9}, \frac{\pi}{3}, \frac{4\pi}{9}$ 7. 562 km. (approx.) 8. 1,135·4 km. (approx.)

9. 4·78 m./s. (approx.) 10. 1 km. (approx.) 11. 1·833 Radian (approx.)
12. 114·59 metre (approx.) 13. 1745 m. (approx.) or 1·75 m. (approx.)

Exercise 8.2

1. (*i*) $\frac{1}{\sqrt{6}}$ (*ii*) $1+\frac{1}{\sqrt{2}}$ 2. $\tan\theta = \frac{3}{4}, \sin\theta = -\frac{3}{5}$

3. $\sin A = -\frac{1}{\sqrt{5}},\ \tan A = -2$ 4. $\sin A = -\frac{\sqrt{3}}{2},\ \tan A = \sqrt{3}$

5. $\sin A = -\frac{-5}{13},\ \cos A = \frac{12}{13}$ 9. $\frac{x^2+y^2}{x^2-y^2}$

12. (*i*) $\frac{27}{4}$ (*ii*) $\frac{17}{12}$ (*iii*) $\frac{3}{4}$ (*iv*) $\frac{5\sqrt{3}}{6}$ 13. 2

Exercise 8.3

7. (*i*) 0 (*ii*) 0 (*iii*) Undefined (*iv*) $\frac{1}{\sqrt{3}}$ (*v*) $\frac{2}{\sqrt{3}}$ (*vi*) Undefined

(*vii*) $-\frac{1}{2}$ (*viii*) $\frac{\sqrt{3}}{2}$

9. (*i*) 0 (*ii*) 1 (*iii*) 2 (*iv*) 2 (*v*) 2

11. (*i*) $\frac{5\pi}{3}$ (*ii*) $\frac{2\pi}{3}$ (*iii*) $\frac{4\pi}{3}$ (*iv*) $\frac{7\pi}{4}$

12. (*i*) $\frac{\pi}{6}$ (*ii*) $\frac{\pi}{3}$ (*iii*) $\frac{\pi}{6}$ (*iv*) $\frac{\pi}{4}$ (*v*) $\frac{\pi}{6}$

13. (*i*) $\frac{2\pi}{3}, \frac{4\pi}{3}$ (*ii*) $\frac{\pi}{6}, \frac{5\pi}{6}$ (*iii*) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(*iv*) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (*v*) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(*vi*) $\frac{\pi}{3}, \frac{5\pi}{3}$ (*vii*) $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

Exercise 9.1

5. (a) x (b) $\frac{\sqrt{a}}{B}$ (c) $\frac{a^2-b^2}{ab}$ (d) 1 (e) 1 (f) $\left(\frac{a}{b}\right)^{a+b}$

8. (a) 1 (b) 0 (c) 3 t 2
9. (a) 0 (b) $x = 1,\ y = 1$ (c) $x = 2,\ y = -2$ (d) $x = -1,\ y = 1$

Exercise 9.2

8. (a) 1.01302 (b) 19995.62 9. (a) 9.2104 (b) –4.90779 (c) 230.76
11. (a) $x = \log(1-y),\ \log a < y < 1$ (b) $x = 10^y,\ -a < y < a$
(c) $x = \sqrt{y},\ 0 < y < a$
12. $D_f = (2, \infty),\ R_f = R$
14. (a) $D_f = [-5, 5],\ R_f = [0, 5]$ (b) $D_f = [-2, 2],\ R_f = [0, 4]$
(c) $D_f = (-5, 5),\ R_f = R$

Exercise 10.1

1. $1 + 5y + 10y^2 + 10y^3 + 5y^4 + y^5$
(i) $1 - 5y + 10y^2 - 10y^3 + 5y^4 - y^5$
(ii) $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$
2. (a) $1 + 4x + 240x^2 + 1280x^3 + \ldots\ldots\ldots\ldots$
(b) $1 - 21x + 189x^2 - 945x^3 + \ldots\ldots\ldots\ldots$
3. (a) $1 + 8x + 28x^4 + 56x^6 + \ldots\ldots\ldots\ldots$ and $1\cdot 082856$
4. (a) $1 - 10x + 40x^2 - \ldots\ldots\ldots\ldots$
(b) $1 + 27x + 324x^2 + \ldots\ldots\ldots\ldots$
(c) $1 + 17x + 94x^2 + \ldots\ldots\ldots\ldots$
5. (a) $1 - 14x^2 + 84x^4 - 280x^6 \ldots\ldots\ldots\ldots$
(b) $1 + \frac{16}{x} + \frac{112}{x^2} + \frac{448}{x^3} \ldots\ldots\ldots\ldots$
(c) $1 - \frac{7}{2}\cdot\frac{1}{x} + \frac{21}{4}\cdot\frac{1}{x^2} - \frac{35}{8}\cdot\frac{1}{x^3} + \ldots\ldots\ldots\ldots$
6. (a) $1 - 6x + 15x^2 - 20x^3 + \ldots\ldots\ldots\ldots$
(b) $1 + 12x + 604x^2 + 160x^3 + \ldots\ldots\ldots\ldots$
(c) $1 + 6x + 3x^2 - 40x^3 + \ldots\ldots\ldots\ldots$

Exercise 10.2

10. (a) $32 + 80x^2 + 80x^4 + 40x^6 + 10x^8 + x^{10}$

(*b*) $64-\frac{96}{x}+\frac{60}{x^2}-\frac{20}{x^3}+\frac{15}{4x^4}-\frac{3}{8x^5}+\frac{1}{64x^6}$

11. (*a*) $64+576x+2160x^2+4320x^3+$..................

(*b*) $1024-\frac{640}{x}+\frac{160}{x^2}-\frac{20}{x^3}+$.....................

12. $p=2,\ r=64,\ s=60$ 13. 7 14. $64+48x+15x^2+\frac{5}{2}x^3+$.................

15. $31\cdot2080$ 16. $n=8$, Number of term 9 and middle term $\frac{35}{128}$

17. $(a)\ x=6$ $(b)\ k=2$

Exercise 11.1

1. $(i)\ \sqrt{13}$ Unit $(ii)\ 4\sqrt{2}$ Unit $(iii)\ (a-b)\sqrt{2}$ Unit (iv) 1 Unit $(v)\ \sqrt{13}$ Unit

5. $k=-5, 5$ 6. $16\cdot971$ (approx.) 9. B is nearer, A is farer

Exercise 11.2

1. (i) 7 Unit, $4\sqrt{2}$ Unit, 5 Unit, $12+4\sqrt{2}$ Unit (ii) 14 sq. Unit

2. (i) 6 sq. Unit (ii) 24 sq. Unit

3. $\sqrt{58}$ Unit, $\sqrt{10}$ Unit, $16\cdot971$ Unit 4. a^2 sq. Unit

5. 10 Unit, 10 Unit, 40 sq. Unit

6. If $a = 5$, $\frac{119}{2}$ sq. Unit

If $a = 15$, $\frac{109}{2}$ sq. Unit

7. $a = 2$, $5\frac{1}{3}$

If $a=2$, ABC is right angled triangle AC is hypotenuse $\angle BAC$ is right angle

8. (i) 21 sq. Unit (ii) 24 sq. Unit (iii) 15 sq. Unit 10. $P=20$

Exercise 11.3

1. (a) -1 (b) $\frac{7}{4}$ (c) 1 (d) 2 2. 5 4. $\frac{1}{2}$ 5. 1, 2

Exercise 11.4

10. $y=2x-5$ 11. $(a)\ y=-x+6$ $(b)\ y=x-3$ $(c)\ y=3x-3a$

12. $(a)\ y = 3x - 5$ $(b)\ y = -3x - 5$ $(c)\ y = 3x + 5$ $(d)\ y = -3x + 5$

13. $(a)\ (1, 0); (0, 3)$ $(b)\ \left(-\frac{6}{5}, 0\right); (0, 3)$ $(c)\ \left(-\frac{4}{3},\ 0\right); (0, -1)$

14. $y = k(x - k); k = 2, 3$ 15. $y = \frac{1}{k}(x + k); k = -1, 2$ 16. $k = \frac{11}{2}$

Exercise 13.2

7. 636 sq. metre, 20.5 metre, 864 cubic metre 8. 1 cubic metre, 7.8 sq. metre, 9. 300 sq. cm. (approx.) 10. 8.75 m, 3.2 m. 11. 188.5 sq. cm. (approx.), 301 cubic cm. (approx.) 12. 25 cm. (approx.) 13. 91.63 cubic cm. (approx.) 14. 452.39 sq. cm. (approx.), 904.8 cubic cm. (approx.)
15. 1 cm. 16. 11.37 cm. (approx.) 17. 1.06 cm. (approx.) 18. 4
19. 1308.82 cubic cm. (approx.) 20. 78.5 sq. cm. (approx.) 21. 7.48 sq. m. (approx.), 22. 83800 23. 2 cm, 12 cm, 12 cm. 24. 2086.49 sq. m. (approx.) 25. 798 sq. cm, 1550 cubic cm. 26. 203.14 sq. cm, 207.85 cubic cm. 27. 298.38 sq. cm, 311.77 cubic cm. 28. 110.85 sq. cm, 60.34 cubic cm. 29. 40.65 sq. cm, 16 cubic cm. 30.4662.86 cubic cm.

Exercise 14

7. (i), $\frac{1}{2}$ (ii), $\frac{7}{30}$ (iii), $\frac{7}{30}$ (iv) $\frac{4}{15}$ 8. $\frac{1}{38}$ 9. $\frac{2}{3}$ 10. $\frac{98}{639}$

11. $(i)\frac{23}{1000}$ (ii) $\frac{1}{400}$ 12. (i) $\frac{157}{1897}$ (ii) $\frac{1630}{1897}$ (iii) $\frac{424}{1897}$

15. (i) $\frac{8}{63}$ (ii) $\frac{25}{63}$ 16 $\frac{4}{45}$

2013

Academic Year

9-10 Higher Math

সমৃদ্ধ বাংলাদেশ গড়ে তোলার জন্য যোগ্যতা অর্জন কর

– মাননীয় প্রধানমন্ত্রী শেখ হাসিনা

জ্ঞান মানুষের অন্তরকে আলোকিত করে

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